Homework 1: Orbit analysis. Fixed and periodic points.

1- Use graphical analysis to describe the fate of all orbits for the next functions.

1. $F(x) = 3x$, $F(x) = \frac{1}{3x}$, $F(x) = -2x + 1$;
2. $F(x) = x^3 - 3x$, $F(x) = |x|$;
3. $F(x) = x - x^2$, $F(x) = sin(x)$;

2- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous functions. Let us assume that there exist two points x_1 and x_2 such that $f(x_1) < x_1$ and $f(x_2) > x_2$. Show that f has a fixed point.
3 Find the set $\{x : F^n(x) \to \pm\infty\}$ for the following functions

1. $T(x) = \begin{cases} 2x \;\; 0 \leq x \leq \frac{1}{2} \\ 2 - 2x \;\; \frac{1}{2} < x \leq 1 \end{cases}$
2. $F(x) = 2x(1 - x)$, $F(x) = 4x(1 - x)$, $F(x) = 8x(1 - x)$.

4 Perform a complete orbit analysis of the following functions:

1. $F(x) = \frac{1}{2}x - 2$, $F(x) = -2x + 1$
2. $F(x) = x^5$, $F(x) = |x|$
3. $F(x) = \exp(x)$, $F(x) = \frac{1}{x}$

5- Let $F_{a,b}(x) = ax + b$ where a and b are constants.

1. Find the fixed points of $F_{a,b}(x)$
2. For which values of a and b the function $F_{a,b}(x)$ has not fixed points?
3. For which values of$a$ and b the function $F_{a,b}(x)$ has only one fixed points?
4. Show that if $0 < |a| < 1$ then $F_{a,b}(x)$ has one fixed points. Show that any positive orbit converges to this fixed point.
5. What is the behavior of any orbit of $F_{a,b}(x)$ if $a = 0$?
6. Show that if $|a| > 1$ then $F_{a,b}(x)$ has one fixed points. Show that any orbit negative converges to this fixed point.

7. What happens if $|a| = 1$?

6- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous increasing function. Show that either one of the following statements holds:

1. for any $x \in \mathbb{R}$ follows that $f^n(x) \to +\infty$;
2. for any $x \in \mathbb{R}$ follows that $f^n(x) \to -\infty$;
3. f has at least one fixed point.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous decreasing function. Show that f has at least one fixed point.

Compare this exercise to exercise 4.

7- For each of the following functions, find all the fixed points and classify them as attracting, repelling, or neutral.

1. $F(x) = x^2 - x/2$, $F(x) = x(1 - x)$
2. $F(x) = 3x(1 - x)$, $F(x) = x^4 - 4x^2 + 2$
3. $F(x) = x^3 - 3x$, $F(x) = -sin(x)$, $F(x) = \frac{\pi}{2} sin(x)$
4. $T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \frac{1}{2} < x \leq 1 \end{cases}$

8- For each of the following functions, zero lies on a periodic orbit. Classify this orbit as attracting, repelling or neutral.

1. $F(x) = 1 - x^2$, $F(x) = \frac{\pi}{2} cos(x)$
2. $F(x) = \frac{-1}{2} x^3 - \frac{3}{2} x^2 + 1$, $F(x) = |x - 2| - 1$
3. $F(x) = \begin{cases} x + 1 & x \leq 3.5 \\ 2x - 8 & x > 3.5 \end{cases}$

9- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous functions. Let p be a fixed point and assume that there exists $\epsilon > 0$ such that for any $x \in (p - \epsilon, p + \epsilon) \setminus \{p\}$ holds that $-x + 2p < f(x) < x$, then it follows that p is an attracting fixed point.

10- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous functions. Let p be a fixed point such that there exists $\epsilon > 0$ such that for any

1. $x \in (p, p + \epsilon)$ holds that $x < f(x)$,
2. $x \in (p - \epsilon, p)$ holds that $f(x) < x$,

then it follows that p is a repelling fixed point.

11- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous functions. Let p be a fixed point such that there exists $\epsilon > 0$ such that for any

1. $x \in (p, p+\epsilon)$ holds that $f(x) < -x + 2p$,

2. $x \in (p-\epsilon, p)$ holds that $-x + 2p < f(x)$,

then it follows that p is a repelling fixed point.
12- Consider the function defined in exercise 6 and try to characterize their periodic points:

1. Let $D : [0,1] \to [0,1]$ be the function defined in the following way:
   $D(x) = \begin{cases} 2x & x \leq 0.5 \\ 2x - 1 & 0.5 < x \leq 1 \end{cases}$ (usually called the doubling function). Suppose that x_0 is a periodic point of D. Evaluate $D^{n\prime}(x_0)$. Is it attracting or repelling?

2. Let $T : [0,1] \to [0,1]$ be the function defined in the following way:
   $T(x) = \begin{cases} 2x & x \leq 0.5 \\ 2 - 2x & 0.5 < x \leq 1 \end{cases}$ (usually called the tent function). Suppose that x_0 is a periodic point of D. Evaluate $D^{n\prime}(x_0)$. Is it attracting or repelling?