Homework 2: Doubling, Tent and expanding functions.

1- Let $D : [0,1] \to [0,1]$ be the doubling function.

1. Try to sketch the graphs of D^2, D^3, and D^n for any positive integer n.
2. How many periodic points of period n has D?
3. For each point $x \in [0,1]$, calculate the cardinal of $D^{-n}(x)$.
4. Is the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lambda x$ chaotic? where λ is a constant that verifies that $|\lambda| > 1$.

2- Let $T : [0,1] \to [0,1]$ be the tent function. Answer the same questions formulated in (**1-**).

3- Let $D : [0,1] \to [0,1]$ be the doubling function.

1. Is it expanding? Justify (prove it).
2. Is it chaotic (expansive)? Justify (prove it).
3. Are the periodic point dense? Justify.
4. Is the pre-orbit of any point dense?.

4- Let $T : [0,1] \to [0,1]$ be the tent function. Answer the same questions formulated in (**3-**).

5- Consider the function $F : \mathbb{C} \to \mathbb{C}$ given by $F(z) = z^2$.

1. Which are the fixed points? Are they repelling or attracting?
2. Try to relate F with the doubling function.
3. Does the map have periodic points of arbitrarily large period?
4. Is it chaotic (expansive)? Justify.
5. Try to describe the dynamic of F.

6- Let $f : [a, b] \to [c, d]$ be a continuous function with continuous derivative and such that $[a, b]$ and $[c, d]$ are two intervals with the property that $[a, b] \subset [c, d]$ and $|f'(x)| > \lambda$ for some constant $\lambda > 1$. Prove that f has a unique repelling fixed point.

7- Let $f : [0, 1] \to [0, 1]$ be a function such that the interval $[0, 1]$ is subdivided in 2 intervals, i.e.: $[0, 1] = [0, c) \cup [c, 1]$ for some $0 < c < 1$ and such that the function f has 2 branches, i.e.: there are two increasing functions f_1 and f_2 with continuous derivative such that

$$f_1 = f_{|[0,c)} : [0, c) \to [0, 1) \quad f_2 = f_{[c,1]} : [c, 1] \to [0, 1].$$

Assume that:

1. $f_1([0, c)) = [0, 1)$ and $f_2([c, 1]) = [0, 1]$;
2. there is a constant $\lambda > 1$ such that $f_1'(x) > \lambda$ for any $x \in [0, c)$ and $f_2'(x) > \lambda$ for any $x \in [c, 1]$.

1. Try to make a model of the graphic of f. Which are the fixed points of f? How many fixed point does f have?
2. Sketch the graphs of $f^2(x)$ and $f^3(x)$. How does the graph of $f^n(x)$ look like? How many fixed points does f^n have for any positive integer n? How many periodic points of period n does f have for any positive integer n? Moreover, prove that the periods of the periodic points are not bounded.
3. Does f has an attracting periodic point? Does f has a neutral periodic point? Are all the periodic points repelling?
4. Prove that the periodic points are dense. Prove that the pre-orbit of any point is dense.
5. Is the map f chaotic? If it is the case, try to prove it.
6. Take a map g C^1-close to f. What can you say about this new map?

8- Let $f : [a, b] \to [a, b]$ be a function such that the following properties hold:

1. the interval $[a, b]$ is subdived in k intervals, i.e.:

$$[a, b] = \cup_{i=1}^{k-1}[a_i, a_{i+1}) \text{ with } a = a_1 < a_2 < ..... < a_k = b$$

2. the function f has k branches, i.e.: for each positive integer i with $1 \leq i \leq k - 1$ follows that there is a function f_i with continuous derivative

$$f_{[a_i,a_{i+1})} = f_i : [a_i, a_{i+1}) \to [a, b]$$

verifying:

(a) $f_i([a_i, a_{i+1})) = [a, b)$

(b) there is a constant $\lambda > 1$ such that $f_i'(x) > \lambda$ for any $x \in [a_i, a_{i+1})$.

Answer all the same question formulated in problem 3.

9- Let $f : [0,1] \to [0,1]$ be a function such that the interval $[0,1]$ is subdivided in 2 intervals, i.e.: $[0,1] = [0,c) \cup [c,1]$ for some $0 < c < 1$ and such that the function f has 2 branches, i.e.: there are two functions f_1 and f_2 with continuous derivative such that f_1 is increasing and f_2 is decreasing,

$$f_1 = f_{|[0,c)} : [0,c) \to [0,1) \quad f_2 = f_{[c,1]} : [c,1] \to [0,1].$$

Assume that:

1. $f_1([0,c)) = [0,1)$ and $f_2([c,1]) = [0,1]$;
2. there is a constant $\lambda > 1$ such that $f_1'(x) > \lambda$ for any $x \in [0,c)$ and $|f_2'(x)| > \lambda$ for any $x \in [c,1]$.

10- Try to generalize the tent map as we have done in the third question with the doubling function.

1. Try to make a model of the graphic of f. Which are the fixed points of f? How many fixed point does f have? How do you relate this map with the tent map?
2. Sketch the graphs of $f^2(x)$ and $f^3(x)$. How does the graph of $f^n(x)$ look like? How many fixed points does f^n have for any positive integer n? How many periodic points of period n does f have for any positive integer n? Moreover, prove that the periods of the periodic points are not bounded.
3. Does f has an attracting periodic point? Does f has a neutral periodic point? Are all the periodic points repelling?
4. Prove that the periodic points are dense. Prove that the pre-orbit of any point is dense.
5. Is the map expansive? Is the map chaotic? If it is the case, try to prove it.
6. Take a map g C^1-close to f. What can you say about this new map?