DISTINGUISHED TAME SUPERCUSPIDAL REPRESENTATIONS

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1. Introduction

In these notes, we describe some examples of distinguished tame supercuspidal representations of reductive p-adic groups. We also discuss some general results about distinguished tame supercuspidal representations from a series of joint papers with Jeffrey Hakim ([HM2]–[HM4]). Finally, we discuss the application of results from [HM4] to obtain a parametrization of equivalence classes of tame supercuspidal representations.

2. Basic notation and definitions

Let F be a nonarchimedean local field. We assume that the residual characteristic p of F is odd. We choose the valuation v_F on F that satisfies $\nu_F(F^\times) = \mathbb{Z}$. Let $\mathfrak{o}_F$ be the ring of integers of F, and let $\mathfrak{p}_F$ be the maximal ideal in $\mathfrak{o}_F$.

If $\mathbf{H}$ is a reductive F-group, we write $H = \mathbf{H}(F)$ for the group of F-rational points of $\mathbf{H}$. We use the notation $\mathcal{B}(H) = \mathcal{B}(\mathbf{H}, F)$ for the extended Bruhat-Tits building of H (which, by definition, is the same as the Bruhat-Tits building $\mathcal{B}(H^o)$ of $H^o = \mathbf{H}^o(F)$).

If H is a subgroup of a group G and π is a representation of G on a complex vector space V, let $\mathrm{Hom}_H(\pi, 1)$ be the space of H-invariant linear functionals on V. The representation π is said to be *H-distinguished* if the space $\mathrm{Hom}_H(\pi, 1)$ is nonzero. Let $V(H)$ be the span of the set $\{\, \pi(h)v - v \mid h \in H,\ v \in V \,\}$. Then π is H-distinguished if and only if $V \neq V(H)$. More generally, if χ is a one-dimensional representation of H, let $\mathrm{Hom}_H(\pi, \chi)$ be the space of linear functionals λ on V such that

$$\lambda(\pi(h)v) = \chi(h)\,\lambda(v), \qquad \forall\ v \in V,\ h \in H.$$

Throughout these notes, we assume that $\mathbf{G}$ is a connected reductive F-group. An *involution* θ of G is an automorphism θ of $\mathbf{G}$ of order two that is defined over F. The group $\mathbf{G}^\theta$ of θ-fixed points in $\mathbf{G}$ is a reductive F-group. We will also use the notation θ for the differential of θ.

Assume that $\mathbf{G}$ splits over a tamely ramified extension of F. By a tame supercuspidal representation of G, we mean one of the irreducible supercuspidal representations of G constructed by J.-K. Yu ([Y]). For particular groups such as general linear groups and some classical groups, tame supercuspidal representations were constructed by others. We do not give a list here, but we remark that Howe ([Ho]) gave a construction of the tame supercuspidal representations of p-adic general linear groups.

Date: July 2008.

In [MP1], Moy and Prasad associated to any point $x \in \mathcal{B}(G)$ a parahoric subgroup $G_{x,0}$ of G, a filtration $\{ G_{x,r} \}_{r\geq 0}$ of the parahoric, and a filtration $\{ \mathfrak{g}_{x,r} \}_{r\in\mathbb{R}}$ of the Lie algebra $\mathfrak{g}$. The indexing of these filtrations depends on a choice of affine roots, hence on a choice of normalization of valuation on F. We use the valuation v_F defined above, and if E is an algebraic extension of F, we take the valuation on E that extends v_F.

If $g \in G$, the notation $\operatorname{Int} g$ will be used to denote the automorphism of G given by conjugation by g. The adjoint representation of G on $\mathfrak{g}$ will be denoted by Ad.

We list a few properties of the Moy-Prasad filtrations: Let $x \in \mathcal{B}(G)$ and r, $s \in \mathbb{R}$. Then:

1. If θ is an automorphism of $\mathbf{G}$ that is defined over F, then $\theta(G_{x,r}) = G_{\theta(x),r}$, $r \geq 0$, and $\theta(\mathfrak{g}_{x,r}) = \mathfrak{g}_{\theta(x),r}$. (Here, the notation θ is also used for the differential of θ, and if $x \in \mathcal{B}(G)$, $\theta(x)$ denotes the image of x under the automorphism of $\mathcal{B}(G)$ induced by θ.)
2. $\operatorname{Int} g(G_{x,r}) = G_{x,r}$ and $\operatorname{Ad} g(\mathfrak{g}_{x,r}) = \mathfrak{g}_{g\cdot x,r}$ for $g \in G$.
3. $[G_{x,r}, G_{x,s}] \subset G_{x,r+s}$ and $[\mathfrak{g}_{x,r}, \mathfrak{g}_{x,s}] \subset \mathfrak{g}_{x,r+s}$.
4. If ϖ is a prime element in F, then $\varpi\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+1}$.

The parahoric subgroup $G_{x,0}$ is a subgroup of the stabilizer of x in G. Hence it follows from the second property above that $\mathfrak{g}_{x,r}$ is $\operatorname{Ad} G_{x,0}$-stable, $r \in \mathbb{R}$, and $G_{x,r}$ is a normal subgroup of $G_{x,0}$, $r \geq 0$.

3. A SIMPLE EXAMPLE

Example 3.1. Let $\varepsilon \in \mathfrak{o}_F^\times$ be a nonsquare, $E = F(\sqrt{\varepsilon})$, and $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $\mathbf{G}$ be the corresponding 2×2 unitary group, with F-rational points

$$G = \{ g \in \mathbf{GL}_2(E) \mid {}^t\bar{g}Jg = J \},$$

where, if $g = (g_{ij})_{1\leq i,j\leq 2}$, then $\bar{g} = (\bar{g}_{ij})_{1\leq i,j\leq 2}$, with $\bar{\alpha}$ denoting the image of $\alpha \in E$ under the nontrivial element of $\mathrm{Gal}(E/F)$.

Let $\mathbf{T}$ be the maximal F-torus in $\mathbf{G}$ having F-rational points

$$T = \left\{ t_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \ \mid \ \alpha,\, \beta \in E,\ \alpha\bar{\beta} = -\bar{\alpha}\beta,\ \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}.$$

The map $t_{\alpha,\beta} \mapsto (\alpha + \beta, \alpha - \beta)$ is an isomorphism of T with $E^1 \times E^1$, where E^1 is the kernel of the norm map $N_{E/F} : E^\times \to F^\times$. Because T is compact, the building $\mathcal{B}(T)$ embeds as a point $\{y\}$ in $\mathcal{B}(G)$. Given a real number r, we let ℓ_r be the unique integer such that $\ell_r < r \leq \ell_r + 1$. The Moy-Prasad filtration groups $G_{y,r}$ and lattices $\mathfrak{g}_{y,r}$ associated to y can be described as follows:

$$G_{y,0} = G \cap GL_2(\mathfrak{o}_E), \quad G_{y,r} = \{ g \in G \mid g - I \in M_{2\times 2}(\mathfrak{p}_E^{\ell_r+1}) \}, \quad r > 0,$$
$$\mathfrak{g}_{y,r} = \mathfrak{g} \cap M_{2\times 2}(\mathfrak{p}_E^{\ell_r+1}), \quad r \in \mathbb{R}.$$

The Lie algebra $\mathfrak{t}$ of T consists of matrices of the form

$$X_{(a,b)} = \begin{pmatrix} a\sqrt{\varepsilon} & b\sqrt{\varepsilon} \\ b\sqrt{\varepsilon} & a\sqrt{\varepsilon} \end{pmatrix}, \qquad a,\, b \in F.$$

If $r \in \mathbb{R}$, let

$$\begin{aligned} &\mathfrak{t}_r = \{\, X_{(a,b)} \in \mathfrak{t} \mid a,\, b \in \mathfrak{p}_F^{\ell_r+1} \,\} = \mathfrak{t} \cap \mathfrak{g}_{y,r} \\ &T_0 = T, \quad T_r = \{\, t_{\alpha,\beta} \in T \mid \alpha - 1,\, \beta \in \mathfrak{p}_E^{\ell_r+1} \,\} = T \cap G_{y,r}, \quad r > 0. \end{aligned}$$

Given $a,\, b \in F$, define $X^*_{(a,b)} \in \mathfrak{t}^*$ by $X^*_{(a,b)}(X_{(c,d)}) = ac + bd$, $c,\, d \in F$. In the context of this particular example, an element $X^*_{(a,b)}$ of $\mathfrak{t}^*$ is G-generic (in the sense of [Y]) if and only if b is nonzero. Fix a positive integer j. Then, if $a,\, b \in F^\times$ satisfy $v_F(a) \geq 2j+1 = v_F(b)$, the linear functional $X^*_{(a,b)} \in \mathfrak{t}^*$ is G-generic of depth $-2j-1$. Fix such an a and b. If $t_{\alpha,\beta} \in T_{2j+1}$, then $(1 - t_{\alpha,\beta})(1 + t_{\alpha,\beta})^{-1} \in \mathfrak{t}_{2j+1}$ and $t_{\alpha,\beta} T_{2j+2} \mapsto (1 - t_{\alpha,\beta})(1 + t_{\alpha,\beta})^{-1} + \mathfrak{t}_{2j+2}$ defines an isomorphism between T_{2j+1}/T_{2j+2} and $\mathfrak{t}_{2j+1}/\mathfrak{t}_{2j+2}$. If ψ is a character of F that is nontrivial on $\mathfrak{o}_F$ and trivial on $\mathfrak{p}_F$, then the map

$$\begin{aligned} t_{\alpha,\beta} T_{2j+2} \mapsto & \psi(X^*_{(a,b)}((1 - t_{\alpha,\beta})(1 + t_{\alpha,\beta})^{-1})) \\ &= \psi((a(1 - \alpha^2 + \beta^2) - 2b\beta)((1+\alpha)^2 - \beta^2)^{-1}) \end{aligned}$$

defines a G-generic character of T_{2j+1} (because this character is realized by the G-generic element $X^*_{(a,b)}$ of depth $-2j-1$).

Let ϕ_0 be a character of T that is trivial on T_{2j+2} and whose restriction to T_{2j+1} agrees with the above character of T_{2j+1}. Let $\vec{\mathbf{G}} = (\mathbf{T}, \mathbf{G})$. If ρ and ϕ_1 are the trivial characters of T and of G, respectively, and $\vec{\phi} = (\phi_0, \phi_1)$, then $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ is an example of a generic cuspidal G-datum. Yu's construction associates to Ψ an irreducible supercuspidal representation $\pi = \pi(\Psi)$ of G. (In fact, for this particular example, $\pi(\Psi)$ is one of the supercuspidal representations constructed by Adler in [A].)

We will describe the construction of π and show that if ϕ_0 is trivial on the subgroup of T consisting of $\{\, t \in T \mid t^2 = 1 \,\}$, then π is distinguished by a particular 2×2 orthogonal group in G. The torus T normalizes $G_{y,r}$ for every $r \geq 0$. (This is a consequence of the fact that $y \in \mathcal{B}(T)$, but can also be verified using the definitions of T and $G_{y,r}$.) Thus each set $TG_{y,r} = \{\, tk \mid t \in T,\ k \in G_{y,r} \,\}$ ($r \geq 0$) is a subgroup of G. The representation π is obtained via compact induction from a one-dimensional representation $\kappa = \kappa(\Psi)$ of the open compact subgroup $K = K(\Psi) := TG_{y,j+1}$. Let

$$\mathfrak{t}^\perp = \left\{ \begin{pmatrix} a\sqrt{\varepsilon} & b\sqrt{\varepsilon} \\ -b\sqrt{\varepsilon} & -a\sqrt{\varepsilon} \end{pmatrix}, \quad a,\, b \in F. \right\}$$

It is easy to check that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp, \quad \mathfrak{g}_{y,r} = \mathfrak{t}_r \oplus (\mathfrak{t}^\perp \cap \mathfrak{g}_{y,r}), \qquad r \in \mathbb{R}.$$

If $r > 0$, the map $k \mapsto (k-1)(k+1)^{-1}$ from $G_{y,r}$ to $\mathfrak{g}_{y,r}$ is a bijection. Define

$$\mathcal{J} = \{\, k \in G_{y,j+1} \mid (k-1)(k+1)^{-1} \in \mathfrak{t}_{2j+2} + (\mathfrak{t}^\perp \cap \mathfrak{g}_{y,j+1}) = \mathfrak{g}_{y,2j+2} + (\mathfrak{t}^\perp \cap \mathfrak{g}_{y,j+1}). \,\}$$

By definition, $G_{y,2j+2} \subset \mathcal{J} \subset G_{y,j+1}$. The map $k\,G_{y,2j+2} \mapsto (k-1)(k+1)^{-1} + \mathfrak{g}_{y,2j+2}$, $k \in G_{y,j+1}$, defines an isomorphism between $G_{y,j+1}/G_{y,2j+2}$ and $\mathfrak{g}_{y,j+1}/\mathfrak{g}_{y,2j+2}$. Under this isomorphism, $\mathcal{J}/G_{y,2j+2}$ corresponds to the subgroup $((\mathfrak{t}^\perp \cap \mathfrak{g}_{y,j+1}) + \mathfrak{g}_{y,2j+2})/\mathfrak{g}_{y,2j+2}$, Thus

$\mathcal{J}$ is a subgroup of $G_{y,j+1}$. Note that $T\cap\mathcal{J}=T_{2j+2}$. Using the fact that both $\mathfrak{t}_{2j+2}$ and $\mathfrak{t}^{\perp}\cap\mathfrak{g}_{y,j+1}$ are $\operatorname{Ad}T$-stable, we can see that T normalizes $\mathcal{J}$. Also, since $G_{y,j+1}=T_{j+1}\mathcal{J}$, we have $K=T\mathcal{J}$. Now we may define a character κ of K by setting κ equal to ϕ_0 on T and letting κ be trivial on $\mathcal{J}$.

We can apply results of Adler ([A]) or Yu ([Y]) to conclude that $\pi=\operatorname{Ind}_K^G\kappa$ is irreducible.

Let θ be the involution of G defined by $\theta(g)=\bar{g}$. Then $G^\theta=G\cap\mathbf{GL}_2(F)$ is the 2×2 orthogonal group defined by the symmetric matrix J:

$$G^\theta=\{\,g\in\mathbf{GL}_2(F)\mid {}^tgJg=J\,\}.$$

Note that $\theta(t_{\alpha,\beta})=t_{\alpha,\beta}^{-1}$ for all $t_{\alpha,\beta}\in T$. In particular, $\theta(T)=T$. We can see from the definitions that

$$\begin{aligned}&\theta(G_{y,r})=G_{y,r},\quad\forall\, r\geq 0,\qquad \theta(\mathfrak{g}_{y,r})=\mathfrak{g}_{y,r},\quad\forall\, r\in\mathbb{R},\\&\theta(\mathfrak{t}^{\perp})=\mathfrak{t}^{\perp},\qquad\theta(\mathcal{J})=\mathcal{J},\qquad\theta(K)=K.\end{aligned}$$

Note that $\phi_0\circ\theta=\phi_0^{-1}$ follows from the fact that θ acts by inversion on T. As κ is trivial on $\mathcal{J}$, this implies that $\kappa\circ\theta=\kappa^{-1}=\widetilde{\kappa}$.

Note that

$$T^\theta=\{\,t_{\alpha,\beta}\in T\mid t_{\alpha,\beta}^2=1\,\}=\{\,\pm 1,\,\pm J\,\}.$$

Let $k\in K^\theta$. Write $k=tg$, with $t\in T$ and $g\in\mathcal{J}$. Then $\theta(g)g^{-1}=t^2\in T\cap\mathcal{J}=T_{2j+2}$. We can use the fact that p is odd to verify that every element in T_{2j+2} is a square in T_{2j+2}. Thus $t^2=t_1^2$ for some $t_1\in T_{2j+2}$. Now we write $k=(tt_1^{-1})(t_1g)$. Observe that $tt_1^{-1}\in T^\theta$ and $t_1g\in\mathcal{J}^\theta$. It follows that $K^\theta=T^\theta\mathcal{J}^\theta$. From this, we see that

$$\kappa\,|\,K^\theta\equiv 1\iff\phi_0\,|\,\{\,\pm 1,\,\pm J\,\}\equiv 1.$$

It is a simple matter to check that ϕ_0 can be chosen so that $\phi_0\,|\,T^\theta\equiv 1$. (Indeed, if ϕ_0 is nontrivial on T^θ, there exists a character χ of T that is trivial on T_1 and $\chi\,|\,T^\theta=\phi_0\,|\,T^\theta$. The character $\phi_0\chi$ is then trivial on T^θ and it is also G-generic, so we could replace ϕ_0 by $\phi_0\chi$.)

From now on, we assume that $\phi_0\,|\,T^\theta$ is trivial. Let λ_0 be any nonzero element of $\operatorname{Hom}_{K^\theta}(\kappa,1)\simeq\mathbb{C}^*$. Below we will use λ_0 to define an element of $\operatorname{Hom}_{G^\theta}(\pi,1)$.

Before doing that, we make some remarks about the contragredient $\widetilde{\pi}$. The dual representation π^* acts on the dual space V^* by $\pi^*(g)\lambda=\lambda\circ\pi(g^{-1})$, $\lambda\in V^*$. In general (if π is not finite-dimensional), π^* is not a smooth representation of G. By restricting π^* to the subspace of smooth vectors in V^* (that is, the subspace consisting of $\lambda\in V^*$ such that $\pi^*(k)\lambda=\lambda$ for all k in some open compact subgroup of G), we obtain a smooth representation $\widetilde{\pi}$ of G, which is called the contragredient of π. In our example, we have $\pi=\operatorname{Ind}_K^G\kappa$, and (because we are using compact induction, both G and K are unimodular, and π is irreducible) we may (and do) identify the contragredient $\widetilde{\pi}$ of π with the representation $\operatorname{Ind}_K^G\widetilde{\kappa}$. (Note that $\widetilde{\kappa}=\kappa^*$ because κ is finite-dimensional.) The space of $\widetilde{V}$ of $\operatorname{Ind}_K^G\widetilde{\kappa}$ is the space of functions $\tilde{f}:G\to\mathbb{C}^*$ such that

(1) $\tilde{f}(kg)=\widetilde{\kappa}(k)\tilde{f}(g)$ for all $k\in K$ and $g\in G$,

(2) The support of $\tilde{f}$ is contained in finitely many right cosets of K in G.

Given $\tilde{f} \in \widetilde{V}$, $(\widetilde{\pi}(g)\tilde{f})(h) = \tilde{f}(hg)$, $g,\, h \in G$. (The space V of π is defined in an analogous way, with the vectors f in V taking values in the space of κ and satisfying $f(kg) = \kappa(k)f(g)$, for $k \in K$ and $g \in G$.) With the above realization of $\widetilde{\pi}$ as $\operatorname{Ind}_K^G \widetilde{\kappa}$, we have a pairing of $\widetilde{V}$ and the space V of π given by

$$\langle\, \tilde{f},\, f\,\rangle = \int_{K\backslash G} \tilde{f}(g)\cdot f(g)\, dg^{\times}, \qquad \tilde{f} \in \widetilde{V},\, f \in V.$$

Here, $\tilde{f}(g)\cdot f(g)$ is the value of the linear functional $\tilde{f}(g)$ on the vector $f(g)$, which, as κ is one-dimensional, can be viewed as the product of two complex numbers, and $dg^{\times}$ is counting measure on the space $K\backslash G$ of right cosets of K in G. (Note that $\tilde{f}(kg)\cdot f(kg) = (\widetilde{\kappa}(k)\tilde{f}(g))\cdot(\kappa(k)f(g)) = \tilde{f}(g)\cdot f(g)$ for all $k \in K$ and $g \in G$.) Thus the element of V^* with which $\tilde{f}$ is identified is given by $f \mapsto \langle\, \tilde{f},\, f\,\rangle$, $f \in V$. With this realization of $\widetilde{V}$, it is convenient to identify the dual V^* of V as the set of functions from G to $\mathbb{C}^*$ that transform under left translation by elements of K in the same way as elements of $\widetilde{V}$, but are not necessarily supported in fintely many right cosets of K in G. For such a $\lambda \in V^*$, the value of λ on a function f in V can be written as

$$\langle\, \lambda,\, f\,\rangle = \int_{K\backslash G} \lambda(g)\cdot f(g)\, dg^{\times},$$

Observe that the integral here is actually a finite sum because of the support properties of f.

Next, we define $\tilde{f}_0 \in \widetilde{V}$ by

$$\tilde{f}_0(g) = \begin{cases} \widetilde{\kappa}(k)\lambda_0 = \lambda_0 \circ \kappa(k^{-1}), & \text{if } g = k \in K,\\ 0, & \text{if } g \notin K. \end{cases}$$

Observe that it follows from the K^θ-invariance of λ_0 that $\widetilde{\pi}(k)\tilde{f}_0 = \tilde{f}_0$ for all $k \in K^\theta$. Define $\lambda \in V^*$ by:

$$\langle\, \lambda,\, f\,\rangle = \int_{G^\theta} \langle \tilde{f}_0,\, \pi(h)f\,\rangle dh, \qquad f \in V.$$

Here, dh is Haar measure on G^θ.

Note that if $\tilde{f} \in \widetilde{V}$ and $f \in V$ are fixed, the function $g \mapsto \varphi(g) := \langle\, \tilde{f},\, \pi(g)f\,\rangle$ from G to $\mathbb{C}$ is a matrix coefficient of π. It is a locally constant function on G, and, because π is supercuspidal, the support of φ is compact. (Notice that the centre Z of G is compact.) In the definition of $\langle\, \lambda,\, f\,\rangle$, we are integrating a matrix coefficient of π over the subgroup G^θ of G, so the integral will converge because of the compact support of the integrand.

Using the fact that $\tilde{f}_0$ is supported on K, we can rewrite $\langle\, \lambda,\, f\,\rangle$ as follows:

$$\begin{aligned}\langle\, \lambda,\, f\,\rangle &= \int_{G^\theta}\int_{K\backslash G} \tilde{f}_0(g)\cdot \pi(h)f)(g)\, dg^\times dh = \int_{G^\theta} \tilde{f}_0(1)\cdot(\pi(h)f)(1)\, dh\\ &= \int_{G^\theta} \lambda_0\cdot f(h)\, dh\end{aligned}$$

Now we can verify that λ is nonzero. Let $f_0 \in V$ be defined by

$$f_0(g) = \begin{cases}\kappa(k), & \text{if } g = k \in K,\\ 0, & \text{if } g \notin K.\end{cases}$$

Then, since λ_0 is K^θ-invariant, $\lambda_0 \cdot f_0(k) = \lambda_0 \cdot f(1) = \lambda_0$ for all $k \in K^\theta$. By definition of f_0, $\lambda_0 \cdot f_0(h) = 0$ if $h \in G^\theta$ and $h \notin K^\theta$. Therefore $\langle\, \lambda,\, f_0\,\rangle$ is the product of λ_0 and the volume of K^θ in G^θ (which is nonzero, as K^θ is an open subgroup of G^θ).

Finally, we see that, because Haar measure on G^θ is invariant under multiplication by elements of G^θ, λ is G^θ-invariant: If $h_1 \in H$, then

$$\langle\, \lambda,\, \pi(h_1)f\,\rangle = \int_{G^\theta} \langle \tilde{f}_0,\, \pi(hh_1)f\,\rangle dh = \int_{G^\theta} \langle \tilde{f}_0,\, \pi(h)f\,\rangle\, dh = \langle\, \lambda,\, f\,\rangle, \qquad f \in V.$$

Thus we have produced a nonzero $\lambda \in \mathrm{Hom}_G(\pi, 1)$ (under the assumption that $\phi_0 \,|\, T^\theta \equiv 1$). We conclude that π is G^θ-distinguished whenever $\phi_0 \,|\, T^\theta \equiv 1$. We will return to this example later.

4. The general setting

Let G be a connected reductive p-adic group. Suppose that K is an open subgroup of G that contains the centre Z of G and is such that K/Z is compact. Assume that κ is an irreducible smooth representation of K and the compactly induced representation $\pi = \mathrm{Ind}_K^G \kappa$ is irreducible. If θ is an involution of G, then we can construct elements of $\mathrm{Hom}_{G^\theta}(\pi, 1)$ from elements of the space $\widetilde{V}$ of $\widetilde{\pi} = \mathrm{Ind}_K^G \widetilde{\kappa}$. Let W be the space of κ and $\widetilde{W} = W^*$ be the space of $\widetilde{\kappa} = \kappa^*$. Let $\langle \cdot, \cdot\rangle_W$ be the pairing of $\widetilde{W}$ with W given by evaluation of an element of $\widetilde{W}$ on a vector in W. Then, as was the case in Example 3.1, we have a pairing of $\widetilde{V}$ with V given by:

$$\langle\, \tilde{f},\, f\,\rangle = \int_{K\backslash G} \langle\, \tilde{f}(g),\, f(g)\,\rangle_W\, dg^\times, \qquad \tilde{f} \in \widetilde{V},\ f \in V.$$

(Here, we have changed our notation slightly to account for the fact that κ is not necessarily one-dimensional, as it was in Example 3.1.) This pairing satisfies $\langle\, \widetilde{\pi}(g)\tilde{f},\, \pi(g)f\,\rangle = \langle\, \tilde{f},\, f\,\rangle$ for all $\tilde{f} \in \widetilde{V}$ and $f \in V$.

Let θ be an involution of G. Note that the centre Z of G is θ-stable. Because π is irreducible, by Schur's Lemma, there exists a quasicharacter χ_π of Z such that $\pi(z)$ is equal to $\chi_\pi(z)$ times the identity operator on V for each $z \in Z$. Observe that if $\chi_\pi \,|\, Z^\theta$ is nontrivial, then π cannot be G^θ-distinguished because any $\lambda \in V^*$ satisfies $\lambda \circ \pi(z) = \chi_\pi(z)\lambda$, for $z \in Z$.

Therefore for the purposes of this discussion, we assume that $\chi_\pi \,|\, Z^\theta$ is trivial. This implies that all matrix coefficients of π are invariant under translation by elements of Z^θ.

Fix an element $\tilde{f} \in \widetilde{V}$. Define

$$\langle\, \lambda_{\tilde{f}},\, f \,\rangle = \int_{G^\theta/Z^\theta} \langle\, \tilde{f},\, \pi(h) f \,\rangle \, dh^\times, \qquad f \in V.$$

Here, $dh^\times$ is Haar measure on G^θ/Z^θ. Note that Z is θ-stable, and the function $h \mapsto \langle\, \tilde{f},\, \pi(h) f \,\rangle$ on G^θ is compactly supported modulo Z^θ, because the function is the restriction to G^θ of a matrix coefficient of the supercuspidal representation π. Because we have assumed that $\chi_\pi \,|\, Z^\theta$ is trivial, the restriction of any matrix coefficient to G^θ is Z^θ-invariant, so the matrix coefficient may be viewed as a function on G^θ/Z^θ. The G^θ-invariance of Haar measure on G^θ/Z^θ implies that $\langle\, \lambda_{\tilde{f}},\, \pi(h_1) f \,\rangle = \langle\, \lambda_{\tilde{f}},\, f \,\rangle$ for all $f \in V$. Thus $\lambda_{\tilde{f}} \in \operatorname{Hom}_{G^\theta}(\pi, 1)$. If $\tilde{f}$ is chosen arbitrarily, $\lambda_{\tilde{f}}$ could be zero. In the example, we chose a particular $\tilde{f}$ which was defined in terms of a nonzero element of $\operatorname{Hom}_{K^\theta}(\kappa, 1)$.

Suppose we take $\tilde{f}$ to be supported on K and nonzero. Then

$$\tilde{f}(g) = \begin{cases} \tilde{\kappa}(k)\tilde{f}(1), & \text{if } g = k \in K, \\ 0, & \text{if } g \notin K. \end{cases}$$

Then

$$\langle\, \lambda_{\tilde{f}},\, f \,\rangle = \int_{G^\theta/Z^\theta} \int_{K\backslash G} \langle\, \tilde{f}(g),\, \pi(h) f(g) \,\rangle_W \, dg^\times dh^\times = \int_{G^\theta/Z^\theta} \langle\, \tilde{f}(1),\, \tilde{f}(h) \,\rangle_W dh^\times.$$

Let $K^\theta = K \cap G^\theta$. If we take $f \in V$ to be supported in K, then

$$\begin{aligned} \langle\, \lambda_{\tilde{f}},\, f \,\rangle &= \int_{K^\theta/Z^\theta} \langle\, \tilde{f}(1),\, f(k) \,\rangle_W dk^\times \\ &= \langle\, \tilde{f}(1),\, \int_{K^\theta/Z^\theta} f(k)\, dk^\times \,\rangle_W \\ &= \langle\, \int_{K^\theta/Z^\theta} \tilde{f}(k)\, dk^\times,\, f(1) \,\rangle_W. \end{aligned}$$

The projection of $f(1)$ onto the K^θ-stable vectors in W is a nonzero scalar multiple of $\int_{K^\theta/Z^\theta} f(k)\, dk^\times$. Suppose that W contains nonzero K^θ-stable vectors. Then we can take $f(1)$ to be one such vector, and we can take $\tilde{f}(1)$ to be a nonzero element of $\operatorname{Hom}_{K^\theta}(\kappa, 1)$ (that is, a K^θ-stable vector in $\widetilde{W}$) that satisfies $\langle\, \tilde{f}(1),\, f(1) \,\rangle_W \neq 0$. With these choices, $\langle\, \lambda_{\tilde{f}},\, f \,\rangle \neq 0$.

More generally, if $g \in G$, and W contains nonzero $K \cap gG^\theta g^{-1}$-invariant vectors, we can choose $\tilde{f} \in \widetilde{W}$ such that $\lambda_{\tilde{f}} \neq 0$, as follows. Let λ_0 be a nonzero element of $\operatorname{Hom}_{K\cap gG^\theta g^{-1}}(\kappa, 1)$.

$$\tilde{f}(g_0) = \begin{cases} \tilde{\kappa}(k)\lambda_0, & \text{if } g_0 g^{-1} = k \in K, \\ 0, & \text{if } g_0 g^{-1} \notin K. \end{cases}$$

Then

$$\langle \lambda_{\tilde f},\, f \rangle = \int_{G^\theta/Z^\theta} \langle \tilde f(g),\, (\pi(h)f)(g) \rangle_W \, dh^\times = \int_{G^\theta/Z^\theta} \langle \lambda_0,\, f(gh) \rangle_W dh^\times, \quad f \in V.$$

If we choose $f \in V$ such that f is supported on Kg and $f(1)$ is a $K \cap gG^\theta g^{-1}$-stable vector in W such that $\langle \lambda_0,\, f(1) \rangle_W \neq 0$, then $\langle \lambda_{\tilde f},\, f \rangle \neq 0$.

Proposition 4.1. [HMa1] *Let $g \in G$. If $\lambda_0 \in \mathrm{Hom}_{K \cap gG^\theta g^{-1}}(\kappa, 1)$, let $\tilde f_0 \in \widetilde V$ be the unique function that is supported in Kg and takes the value $\tilde f_0(g) = \lambda_0$. Then*

1. *$\lambda_{\tilde f_0} \in \mathrm{Hom}_{G^\theta}(\pi, 1)$.*
2. *Fix $g \in G$. Let $V_g = \{\, f \in V \mid \mathrm{supp}(f) \subset KgG^\theta \,\}$. The map $\lambda_0 \mapsto \lambda_{\tilde f_0} \,|\, V_g$ is an isomorphism between $\mathrm{Hom}_{K \cap gG^\theta g^{-1}}(\kappa, 1)$ and the space of G^θ-invariant linear functionals on the space V_g.*
3. *Let g range over a set of representatives for the K-G^θ double cosets in G. Then*

$$\mathrm{Hom}_{G^\theta}(\pi, 1) \simeq \bigoplus_{g \in K\backslash G/G^\theta} \mathrm{Hom}_{K \cap gG^\theta g^{-1}}(\kappa, 1).$$

This proposition does not require that π be tame.

Let $g \in G$. Note that $gG^\theta g^{-1} = G^{g\cdot\theta}$, where $g \cdot \theta$ is the involution of G defined by $(g \cdot \theta)(g_0) = g\theta(g)^{-1}\theta(g_0)\theta(g)g^{-1}$, $g_0 \in G$. In this way, we define an action of G on the set of involutions of G. Clearly the map $\lambda \mapsto \lambda \circ \pi(g^{-1})$ defines an isomorphism between $\mathrm{Hom}_{G^\theta}(\pi, 1)$ and $\mathrm{Hom}_{G^{g\cdot\theta}}(\pi, 1)$. The proposition describes $\mathrm{Hom}_{G^\theta}(\pi, 1)$ in terms of the direct sum of the various spaces $\mathrm{Hom}_{K \cap G^{\dot\theta}}(\kappa, 1)$, as $\dot\theta$-ranges over the involutions in the G-orbit of θ that correspond to the distinct K-G^θ double cosets via $KgG^\theta \mapsto g \cdot \theta$.

We remark that, because $\mathrm{Hom}_{K \cap gG^\theta g^{-1}}(\kappa, 1)$ is isomorphic to $\mathrm{Hom}_{K \cap gG^\theta g^{-1}}(\widetilde\kappa, 1)$, the proposition implies that $\mathrm{Hom}_{G^\theta}(\pi, 1)$ is isomorphic to $\mathrm{Hom}_{G^\theta}(\widetilde\pi, 1)$.

5. Returning to Example 3.1

We have seen that whenever $\phi_0 \,|\, T^\theta$ is trivial, then $\mathrm{Hom}_{K^\theta}(\kappa, 1)$ is one-dimensional, and is isomorphic to a one-dimensional subspace of $\mathrm{Hom}_{G^\theta}(\pi, 1)$. To describe $\mathrm{Hom}_{G^\theta}(\pi, 1)$, we need to describe $\mathrm{Hom}_{K \cap G^{g\cdot\theta}}(\kappa, 1)$ for all double cosets KgG^θ. Because κ is one-dimensional in the example, this amounts to determining which double cosets KgG^θ have the property that $\kappa \,|\, K \cap G^{g\cdot\theta}$ is trivial.

Some results from joint work with J. Hakim ([HM4]), which will be discussed in a more general setting later in these notes, give criteria which can be used to determine which double cosets KgG^θ contribute to $\mathrm{Hom}_{G^\theta}(\pi, 1)$. For this particular example, because the cuspidal G-datum $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec\phi)$ which gave rise to π has the form $\vec{\mathbf{G}} = (\mathbf{T}, \mathbf{G})$, $\vec\phi = (\phi_0, 1)$, and $\rho = 1$, and we already know that $\theta(T) = T$, $\theta(K) = K$, and $\mathrm{Hom}_{K^\theta}(\kappa, 1) \simeq \mathrm{Hom}_{T^\theta}(\phi_0, 1)$, the results of [HM4] tell us that a particular K-G^θ double coset contributes to $\mathrm{Hom}_{G^\theta}(\pi, 1)$ if and only if it contains an element g such that $g\theta(g)^{-1} \in T$. Notice that for such a g, $g \cdot \theta \,|\, T = \theta \,|\, T$, so $\phi_0 \,|\, T^\theta \equiv 1$ implies $\phi_0 \,|\, T^{g\cdot\theta} \equiv 1$.

So the next step is to determine which K-G^θ-double cosets in G contain an element g such that $g\theta(g)^{-1} \in T$. Because the set of squares in T coincides with the set $\{\, t\theta(t)^{-1} \mid t \in T \,\}$, if $g \in G$ is such that $g\theta(g)^{-1}$ is a square in T, then $g \in TG^\theta \subset KG^\theta$. Suppose that $t_{\alpha,\beta} \in T$ is equal to $g\theta(g)^{-1}$ for some $g \in G$. We can check that the determinant of $g\theta(g)^{-1}$ is a square in E^1. Thus, $\det t_{\alpha,\beta} = \alpha^2 - \beta^2$ must be a square in E^1. If both $\alpha + \beta$ and $\alpha - \beta$ are squares in E^1, then $t_{\alpha,\beta}$ is a square in T, which implies $g \in KG^\theta$. It follows that if $g \notin KG^\theta$, then neither $\alpha - \beta$ nor $\alpha + \beta$ is a square in E^1. Assume that this is the case. Let $\omega \in \mathfrak{o}_E^\times$ be such that $\omega\bar{\omega}^{-1}$ is a nonsquare in E^1. Then $\alpha - \beta = \omega\bar{\omega}^{-1}\gamma^2$ and $\alpha + \beta = \omega\bar{\omega}^{-1}\gamma^2\delta^2$ for some γ, $\delta \in E^1$. This implies that $t_{\alpha,\beta} = t_{\omega\bar{\omega}^{-1},0}t^2$, where $t \in T$ is the element corresponding to $(\gamma^2\delta^2, \gamma^2) \in E^1 \times E^1$. That is, $t_{\alpha,\beta} = g'\theta(g')^{-1}$ where $g' = t \left(\begin{smallmatrix} \omega & 0 \\ 0 & \bar{\omega}^{-1} \end{smallmatrix}\right)$. This implies that $g \in T \left(\begin{smallmatrix} \omega & 0 \\ 0 & \bar{\omega}^{-1} \end{smallmatrix}\right) G^\theta \subset K \left(\begin{smallmatrix} \omega & 0 \\ 0 & \bar{\omega}^{-1} \end{smallmatrix}\right) G^\theta$. This shows that there are exactly two K-G^θ double cosets in G containing elements g with $g\theta(g)^{-1} \in T$. This allows us to conclude that, for $\pi = \pi(\Psi)$,

$$\dim \operatorname{Hom}_{G^\theta}(\pi, 1) = \begin{cases} 2, & \text{if } \phi_0 \,|\, \langle -1, J \rangle = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that ϕ_0 is chosen to have this property. We remark that for $g = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \bar{\omega}^{-1} \end{smallmatrix}\right)$, we have $g\theta(g)^{-1} \in Z$. Thus $g \cdot \theta = \theta$. As observed above, $g \notin KG^\theta$. Note that $g\theta(g)^{-1} \in Z$ implies that g normalizes G^θ.

6. DEPTH ZERO SUPERCUSPIDAL REPRESENTATIONS

Let π be an irreducible smooth representation of G in a (complex) vector space V. Then π has depth zero (in the sense of Moy and Prasad ([MP1])) if there exists $x \in \mathcal{B}(G)$ such that

$$V^{G_{x,0^+}} = \{\, v \in V \mid \pi(k)v = v \;\forall\, k \in G_{x,0^+} \,\} \neq 0.$$

Here, $G_{x,0^+}$ is the pro-unipotent radical of the parahoric subgroup $G_{x,0}$. If π is supercuspidal and depth zero, then (see [MP2],[Mor]), the following additional conditions hold:

1. For x as above, $G_{x,0}$ is a maximal parahoric subgroup of G.
2. There exists an irreducible smooth representation ρ of $K = N_G(G_{x,0})$ (the normalizer of $G_{x,0}$ in G) such that
   - (a) $\rho \,|\, G_{x,0^+}$ is a multiple of the trivial representation of $G_{x,0^+}$.
   - (b) $\rho \,|\, G_{x,0}$ contains a representation of $G_{x,0}$ that factors to an irreducible cuspidal representation of $G_{x,0}/G_{x,0^+}$.
   - (c) $\pi \simeq \operatorname{Ind}_K^G \rho$.

Remark 6.1. Let $\mathfrak{f} = \mathfrak{o}_F/\mathfrak{p}_F$. The group $G_{x,0^+}$ is a normal subgroup of $G_{x,0}$ and $G_{x,0}/G_{x,0^+}$ is the $\mathfrak{f}$-rational points of a connected reductive $\mathfrak{f}$-group. An irreducible representation of $G_{x,0}/G_{x,0^+}$ is said to be cuspidal if it has no nonzero vectors that are fixed by the unipotent radical of any proper parabolic subgroup of $G_{x,0}/G_{x,0^+}$. Because $G_{x,0}$ is a maximal parahoric subgroup, the group K is compact modulo the centre of G.

If $G = GL_n(F)$ and $G_{x,0}$ is a maximal parahoric subgroup of G, then $G_{x,0}$ is conjugate in G to the group $GL_n(\mathfrak{o}_F) = \{\, g \in G \mid g \in M_{n\times n}(\mathfrak{o}_F),\ \det g \in \mathfrak{o}_F^\times \,\}$ and $G_{x,0}/G_{x,0^+}$ is isomorphic to $GL_n(\mathfrak{f})$.

If $G = SL_2(F)$, there are two G-conjugacy classes of maximal parahoric subgroups of G. Let ϖ be a prime element in F (that is, a generator of the ideal $\mathfrak{p}_F$). One class of maximal parahorics is the set of conjugates of $SL_2(\mathfrak{o}_F) = SL_2(F) \cap GL_2(\mathfrak{o}_F)$. The other is the class of $\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right) SL_2(\mathfrak{o}_F) \left(\begin{smallmatrix}\varpi^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)$.

In general, there are finitely many conjugacy classes of maximal parahoric subgroups in G. These conjugacy classes correspond to the G-orbits of facets of minimal dimension in $\mathcal{B}(G)$. Given two nonconjugate maximal parahoric subgroups $G_{x,0}$ and $G_{y,0}$, the quotients $G_{x,0}/G_{x,0^+}$ and $G_{y,0}/G_{y,0^+}$ are not necessarily isomorphic.

Let π be an depth zero (irreducible) superucuspidal representation of G, and let x, ρ and K be as above. Let $\vec{\mathbf{G}} = (\mathbf{G})$ and $\vec{\phi} = (\phi_0)$, where ϕ_0 is the trivial character of G. Set $\Psi = (\vec{\mathbf{G}}, x, \rho, \vec{\phi})$. Then Ψ is a cuspidal G-datum and $\pi \simeq \pi(\Psi) = \operatorname{Ind}_K^G \rho$ is the corresponding tame superucuspidal representation of G.

Let $G = GL_n(F)$. The centre of G is isomorphic to $F^\times$ (where an element of $F^\times$ is identified with the corresponding scalar mutliple of the identity matrix). The group $K = N_G(GL_n(\mathfrak{o}_F))$ is equal to $F^\times GL_n(\mathfrak{o}_F)$, and the pro-unipotent radical of $GL_n(\mathfrak{o}_F)$ is $\{\, g \in G \mid g - 1 \in M_{n\times n}(\mathfrak{p}_F) \,\}$. If ρ is an irreducible smooth representation of K such that $\rho \,|\, GL_n(\mathfrak{o}_F)$ factors to an irreducible cuspidal representation of $GL_n(\mathfrak{f})$, then $\pi = \operatorname{Ind}_K^G \rho$ is an depth zero (irreducible) supercuspidal representation of G. (All depth zero supercuspidal representations of $GL_n(F)$ have this form (for some choice of ρ).) Note that $F^\times \cap GL_n(\mathfrak{o}_F) \simeq \mathfrak{o}_F^\times$, so ρ is determined by $\rho \,|\, GL_n(\mathfrak{o}_F)$ and the nonzero complex number $\rho(\varpi \cdot 1)$ (since $F^\times \simeq \langle \varpi \rangle \times \mathfrak{o}_F^\times$).

7. DISTINGUISHED DEPTH ZERO SUPERCUSPIDAL REPRESENTATIONS

Let $\pi = \operatorname{Ind}_K^G \rho$ be a depth zero (irreducible) supercuspidal representation of G (with $K = N_G(G_{x,0})$, $G_{x,0}$ a maximal parahoric subgroup of G, and ρ as described in the previous section).

Let θ be an involution of G. According to results of [HM4], if $\operatorname{Hom}_{G^\theta}(\pi, 1)$ is nonzero, there exists $g \in G$ such that $(g \cdot \theta)(K) = K$ and $\operatorname{Hom}_{K^{g\cdot\theta}}(\rho, 1)$ is nonzero. The condition $(g \cdot \theta)(K) = K$ is equivalent to $(g \cdot \theta)(G_{x,0}) = G_{x,0}$, which is easily seen to be equivalent to $\theta(g^{-1} G_{x,0} g) = g^{-1} G_{x,0} g$, that is, $\theta(G_{g^{-1}x,0}) = G_{g^{-1}x,0}$. In this case, ${}^gK = g^{-1}Kg$ is the normalizer of the maximal parahoric subgroup $G_{g^{-1}x,0}$. Let ${}^g\rho$ be the representation of ${}^gK$ defined by $({}^g\rho)(gkg^{-1}) = \rho(k)$, $k \in K$. Then $\pi \simeq \operatorname{Ind}_{{}^gK}^G {}^g\rho$, and we have $\operatorname{Hom}_{K^{g\cdot\theta}}(\rho, 1) \simeq \operatorname{Hom}_{({}^gK)^\theta}({}^g\rho, 1)$. Therefore, after replacing K and ρ by ${}^gK$ and ${}^g\rho$, we may assume that $\theta(K) = K$ and $\operatorname{Hom}_{K^\theta}(\rho, 1) \neq 0$ whenever $\operatorname{Hom}_{G^\theta}(\pi, 1) \neq 0$.

According to Proposition 4.1,

$$\operatorname{Hom}_{G^\theta}(\pi, 1) \simeq \bigoplus_{g \in K\backslash G/G^\theta} \operatorname{Hom}_{K \cap G^{g\cdot\theta}}(\rho, 1).$$

Results of [HM4] give information that determines which terms on the right side of the above equality that could possibly yield a nonzero contribution to $\mathrm{Hom}_{G^\theta}(\pi, 1)$. As mentioned above, if no conjugate of K is θ-stable, then $\mathrm{Hom}_{G^\theta}(\pi, 1) = 0$. Therefore we assume $\theta(K) = K$. Define a map $\tau : G \to G$ by $\tau(g) = g\theta(g)^{-1}$. Note that the property $\theta(K) = K$ implies that either $\tau(KgG^\theta) \subset K$ or $\tau(KgG^\theta) \cap K = \emptyset$. As shown in [HM4], $\tau(KgG^\theta) \cap K = \emptyset$, then $\mathrm{Hom}_{K \cap G^{g\cdot\theta}}(\rho, 1) = 0$. Also, there are only finitely many double cosets KgG^θ such that $\tau(KgG^\theta) \subset K$. Note that for each representative g of each such coset, we have $(g\cdot\theta)(K) = K$.

Let $g_1 = 1, g_2, \ldots, g_m$ be representatives for the distinct K-G^θ double cosets whose images under the map τ lie in K. Then

$$\mathrm{Hom}_{G^\theta}(\pi, 1) \simeq \bigoplus_{1 \le i \le m} \mathrm{Hom}_{K^{g_i\cdot\theta}}(\rho, 1).$$

The results of [HM4] don't give information about $\mathrm{Hom}_{K^{g_i\cdot\theta}}(\rho, 1)$, $1 \le i \le m$. There is some information for $G = GL_n(F)$ in other papers (see [HMa1],[HMa2],[HM2],[HM3]), and we will discuss an example shortly.

Let $\bar{\theta}_i$ be the automorphism of $G_{x,0}/G_{x,0^+}$ defined by $\bar{\theta}_i(kG_{x,0^+}) = (g_i\cdot\theta)(k)G_{x,0^+}$, $k \in G_{x,0}$. Suppose that $\mathrm{Hom}_{K^{g_i\cdot\theta}}(\rho, 1) \neq 0$. Then there is an irreducible cuspidal representation $\bar{\rho}$ of $G_{x,0}/G_{x,0^+}$ such that the representation $k \mapsto \bar{\rho}(kG_{x,0^+})$ of $G_{x,0}$ is a constituent of $\rho \,|\, G_{x,0}$, and

$$\mathrm{Hom}_{(G_{x,0}/G_{x,0^+})^{\bar{\theta}_i}}(\bar{\rho}, 1) \neq 0.$$

This shows that the study of distinguished depth zero supercuspidal representations is closely related to the study of distinguished cuspidal representations of finite groups of Lie type.

The group $\mathcal{G} = G_{x,0}/G_{x,0^+}$ is the $\mathfrak{f}$-rational points of a connected reductive $\mathfrak{f}$-group. Suppose that $\bar{\theta}$ is an involution of $\mathcal{G}$. For particular kinds of cuspidal representations, namely those arising via "Deligne-Lusztig induction", results of Lusztig ([L]) give a formula for the dimension of $\mathrm{Hom}_{\mathcal{G}^{\bar{\theta}}}(\bar{\rho}, 1)$. If $\bar{\rho}$ arises in this way, there exists an elliptic maximal torus $\mathcal{T} \subset \mathcal{G}$ and a character χ of $\mathcal{T}$ that is in "general position" (this is a kind of regularity condition) that give rise to the representation $\bar{\rho}$. Assuming that $\mathcal{T}$ is $\bar{\theta}$-stable, let S be the set of $\mathcal{G}^{\bar{\theta}}$-orbits of tori in $\mathcal{G}$ that are $\mathcal{G}$-conjugate to $\mathcal{T}$ and $\bar{\theta}$-stable. (Note that $g^{-1}\mathcal{T}g$ is $\bar{\theta}$-stable if and only if $g\bar{\theta}(g)^{-1}$ belongs to the normalizer of $\mathcal{T}$ in $\mathcal{G}$.) Lusztig's formula for $\dim \mathrm{Hom}_{\mathcal{G}^{\bar{\theta}}}(\bar{\rho}, 1)$ assigns to each of the $\mathcal{G}^{\bar{\theta}}$-orbits in S a number in $\{-1, 0, 1\}$. The sum of the numbers attached to the various orbits is equal to the desired dimension. To evaluate Lusztig's formula, it is necessary to determine the set S and to compute the number attached to each orbit in S.

If $\mathcal{G}$ is a general linear group, then all irreducible cuspidal representations of $\mathcal{G}$ are obtained from the Deligne-Lusztig construction. Otherwise, that is, of $\mathcal{G}$ is not a general linear group, then some irreducible cuspidal representations of $\mathcal{G}$ are not obtained using the Deligne-Lusztig construction. We will not discuss the details of the Deligne-Lusztig construction here.

8. A depth zero example

Example 8.1. Let n be an even integer and let F' be a quadratic totally ramified extension of F. There exists a prime element $\varpi \in F$ such that $F' = F(\sqrt{\varpi})$. Let $\mathbf{G} = R_{F'/F}GL_n$, where $R_{F'/F}$ is restriction of scalars. Then $G = \mathbf{G}(F) = (R_{F'/F}GL_n)(F) \simeq GL_n(F')$. If $g \in G$, let $\bar{g}$ be the element of G obtained by letting the nontrivial element of $\mathrm{Gal}(F'/F)$ act on the matrix entries of g. Define $\theta(g) = {}^t\bar{g}^{-1}$, $g \in G$. Then θ is an involution of G and $G^\theta = \{\, g \in G \mid g{}^t\bar{g} = 1 \,\}$ is a unitary group.

Let $\mathfrak{o}_{F'}$ be the ring of integers in F'. The subgroup $GL_n(\mathfrak{o}_{F'})$ of G ia a θ-stable maximal parahoric subgroup of G. Also, since F'/F is totally ramified, the residue fields $\mathfrak{f}' = \mathfrak{o}_{F'}/\mathfrak{p}_{F'}$ and $\mathfrak{f} = \mathfrak{o}_F/\mathfrak{p}_F$ are isomorphic. Thus the quotient of $GL_n(\mathfrak{o}_{F'})$ by its pro-unipotent radical is isomorphic to $GL_n(\mathfrak{f})$. The involution $\bar{\theta}$ of $GL_n(\mathfrak{f})$ determined by θ is $\bar{\theta}(g) = {}^tg^{-1}$, $g \in GL_n(\mathfrak{f})$. Thus $GL_n(\mathfrak{f})^{\bar{\theta}}$ is (the $\mathfrak{f}$-rational points of) an orthogonal group.

Let $\mathfrak{f}_n$ be the extension of $\mathfrak{f}$ of degree n. Then, via a choice of basis of $\mathfrak{f}_n$ over $\mathfrak{f}$, we embed $\mathcal{T} = \mathfrak{f}_n^\times$ as a subgroup of $GL_n(\mathfrak{f})$, and we have $\mathcal{T}$ is the $\mathfrak{f}$-rational points of the group $R_{\mathfrak{f}_n/\mathfrak{f}}GL_1$. It is possible to show that the basis of $\mathfrak{f}_n$ over $\mathfrak{f}$ can be chosen so that $\mathcal{T}$ is $\bar{\theta}$-stable and $\bar{\theta}(t) = t^{-1}$ for $t \in \mathcal{T}$. (We will assume that the basis has been chosen this way.) Let χ be a character of $\mathcal{T} \simeq \mathfrak{f}_n^\times$ which satisifies $\chi \circ \sigma \neq \chi$ for every nontrivial element σ in the Galois group $\mathrm{Gal}(\mathfrak{f}_n/\mathfrak{f})$ of $\mathfrak{f}_n$ over $\mathfrak{f}$. Then, via the construction of Deligne and Lusztig, the $\mathrm{Gal}(\mathfrak{f}_n/\mathfrak{f})$-orbit of χ determines the equivalence class of an irreducible cuspidal representation $\bar{\rho}$ of $GL_n(\mathfrak{f})$. Note that $-1 \in \mathfrak{f}_n^\times$ belongs to the centre of $GL_n(\mathfrak{f})$ and is $\bar{\theta}$-stable. Therefore, if $\mathrm{Hom}_{GL_n(\mathfrak{f})^{\bar{\theta}}}(\bar{\rho}, 1)$ is nonzero, the central character of $\bar{\rho}$ must be trivial on -1. The central character is given by $\chi \,|\, \mathfrak{f}^\times$, so we must have $\chi(-1) = 1$ (note that this is equivalent to $\chi \,|\, \mathcal{T}^{\bar{\theta}} \equiv 1$) whenever $\mathrm{Hom}_{GL_n(\mathfrak{f})^{\bar{\theta}}}(\bar{\rho}, 1)$ is nonzero.

Hakim and Mao ([HMa1]) evaluated Lusztig's formula for $\dim \mathrm{Hom}_{GL_n(\mathfrak{f})^{\bar{\theta}}}(\bar{\rho}, 1)$. It turns out that $\mathrm{Hom}_{GL_n(\mathfrak{f})^{\bar{\theta}}}(\bar{\rho}, 1)$ is nonzero if and only if $\chi(-1) = 1$, and in that case, the dimension is equal to one.

Now we turn to constructing a depth zero irreducible supercuspidal representation of G from the representation $\bar{\rho}$. Note that $\sqrt{\varpi}$ is a prime element in F'. The inducing subgroup is

$$K = N_G(GL_n(\mathfrak{o}_{F'})) = F'^\times GL_n(\mathfrak{o}_{F'}) \simeq \langle \sqrt{\varpi} \rangle \times GL_n(\mathfrak{o}_{F'}).$$

Let ρ be a representation of K such that $\rho \,|\, GL_n(\mathfrak{o}_{F'})$ factors to the representation $\bar{\rho}$ of $GL_n(\mathfrak{f})$. (To extend ρ from $GL_n(\mathfrak{o}_{F'})$ to K, we simply set $\rho(\sqrt{\varpi})$ equal to a nonzero complex number.) The representation $\pi = \mathrm{Ind}_K^G \rho$ is (irreducible) supercuspidal.

The action of θ on the centre $F'^\times$ of G is given by $\theta(\gamma) = \bar{\gamma}^{-1}$, $\gamma \in F'^\times$. Thus $(F'^\times)^\theta$ is equal to the kernel F'^1 of the norm map $N_{F'/F}$ from $F'^\times$ to $F^\times$. In particular, $(F'^\times)^\theta \subset GL_n(\mathfrak{o}_{F'})^\theta$, It follows that $K^\theta = GL_n(\mathfrak{o}_{F'})^\theta$. Therefore

$$\mathrm{Hom}_{K^\theta}(\rho, 1) = \mathrm{Hom}_{GL_n(\mathfrak{o}_{F'})^\theta}(\rho, 1) \simeq \mathrm{Hom}_{GL_n(\mathfrak{f})^{\bar{\theta}}}(\bar{\rho}, 1).$$

By remarks above, we conclude that

$$\mathrm{Hom}_{K^\theta}(\rho, 1) \neq 0 \iff \chi(-1) = 1,$$

and $\dim \operatorname{Hom}_{K^\theta}(\rho, 1) \leq 1$.

Hakim and Mao ([HMa2]) describe coset representatives for the $GL_n(\mathfrak{o}_{F'})$-G^θ double cosets in G. Let $\mathcal{A} = \{\, \alpha = (\, \alpha_1, \ldots, \alpha_m \,) \,\}$ satisfying $\alpha_i = (a_i, b_i, \epsilon_i)$, with

1. $a_i \in \mathbb{Z}$, $a_1 > \cdots > a_m$.
2. $b_1 + \cdots + b_m = n$, $b_j \in \mathbb{Z}$, $b_j > 0$.
3. If a_i is odd, then b_i is even and $\epsilon_i = 1$.
4. If a_i is even, then $\epsilon \in \{\, 1, \delta \,\}$, where $\delta \in \mathfrak{o}_{F'}^\times$ is a nonsquare.

If a_i is odd, let ϖ^{α_i} be the $b_i \times b_i$ matrix equal to $\sqrt{\varpi}^{a_i}$ times the matrix with $b_i/2$ 2×2 blocks of the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ on the diagonal, and zeros elsewhere. If a_i is odd, let ϖ^{α_i} be $b_i \times b_i$ diagonal matrix $\sqrt{\varpi}^{a_i} \operatorname{diag}(1, 1, 1, \ldots, 1, \epsilon_i)$. Finally, let h_α be the block diagonal matrix with diagonal blocks $\varpi^{\alpha_1}, \ldots, \varpi^{\alpha_m}$, respectively. Let $g_\alpha \in G$ be such that $g_\alpha \theta(g_\alpha)^{-1} = h_\alpha$ (such a g_α always exists). The set $\{\, g_\alpha \mid \alpha \in \mathcal{A} \,\}$ parametrizes the coset space $GL_n(\mathfrak{o}_{F'}) \backslash G / G^\theta$.

As shown in [HM4] in general, and [HM2] for this example, if $\theta(K) = K$, then whenever $\operatorname{Hom}_{K \cap G^{g \cdot \theta}}(\rho, 1)$ is nonzero, we must have $\tau(g) = g\theta(g)^{-1} \in K$. The basic idea of the proof of this fact is as follows. It $\tau(g) \notin K$, then it is possible to show that $GL_n(\mathfrak{o}_{F'}) \cap G^{g \cdot \theta}$ contains a set that projects (under the map $GL_n(\mathfrak{o}_{F'}) \to GL_n(\mathfrak{f})$) to to the unipotent radical N of a proper parabolic subgroup of $GL_n(\mathfrak{f})$. If we have $\operatorname{Hom}_{K \cap G^{g\dot{\theta}}}(\rho, 1)$ nonzero, then $\operatorname{Hom}_N(\bar{\rho}, 1) \neq 0$. This is equivalent to existence of nonzero vectors in the space of $\bar{\rho}$ that are fixed under N. This contradicts cuspidality of the representation $\bar{\rho}$.

We can easily check that $h_\alpha = \tau(g_\alpha) \notin K$ whenever $\alpha \in \mathcal{A}$ satisfies $m > 1$. Thus, we need only consider those $\alpha \in \mathcal{A}$ with $m = 1$. The case $g_\alpha \in KG^\theta$ has already been taken care of, as we have already analyzed $\operatorname{Hom}_{K^\theta}(\rho, 1)$. If $m = 1$ and $g_\alpha \notin KG^\theta$, then $\alpha = (a, n, 1)$ for some odd integer a, so h_α is equal to $\sqrt{\varpi}^a$ times the $n \times n$ block diagonal matrix with $n/2$ copies of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ as diagonal blocks. Furthermore, $\alpha = (a, n, 1)$ and a is odd, then

$$GL_n(\mathfrak{o}_{F'}) g_\alpha G^\theta \subset K g(1, n, 1) G^\theta.$$

That is,

$$\bigcup_{\alpha = (a,n,1),\, a \text{ odd}} GL_n(\mathfrak{o}_{F'}) g_\alpha G^\theta = K g_{1,n,1} G^\theta.$$

Let $\dot{\theta} = g_{(1,n,1)} \cdot \theta$. We can now conclude that

$$\operatorname{Hom}_{G^\theta}(\pi, 1) \simeq \operatorname{Hom}_{K^\theta}(\rho, 1) \oplus \operatorname{Hom}_{K^{\dot{\theta}}}(\rho, 1).$$

To complete the example, we need to determine $\operatorname{Hom}_{K^{\dot{\theta}}}(\rho, 1)$.

Let $g = g_{(1,n,1)}$. Note that $\tau(g) = \sqrt{\varpi} J$, where J is skew-symmetric and invertible. For $g_0 \in G$, $\dot{\theta}(g_0) = J^t \bar{g}_0^{-1} J^{-1}$. Since $J \in GL_n(\mathfrak{o}_{F'})$, and the image of J in $GL_n(\mathfrak{f})$ is skew-symmetric, we find that

$$\dot{\bar{\theta}}(x) = J \bar{\theta}(x) J^{-1} = J^t x^{-1} J^{-1},$$

for $x \in GL_n(\mathfrak{f})$. Therefore $GL_n(\mathfrak{f})^{\dot{\bar{\theta}}} \simeq Sp_n(\mathfrak{f})$. If $\operatorname{Hom}_{K^{\dot{\theta}}}(\rho, 1)$ is nonzero, then $\operatorname{Hom}_{GL_n(\mathfrak{f})^{\dot{\bar{\theta}}}}(\bar{\rho}, 1)$ is nonzero. But results for Heumos and Rallis ([HR]) show that (for n even), an irreducible cuspidal representation of $GL_n(\mathfrak{f})$ cannot be distinguished by $Sp_n(\mathfrak{f})$. Thus $\mathrm{Hom}_{K^{\dot\theta}}(\rho, 1)$ is zero.

We can now conclude that $\mathrm{Hom}_{G^\theta}(\pi, 1) \simeq \mathrm{Hom}_{K^\theta}(\rho, 1)$ has dimension at most one and is nonzero if and only if $\chi(-1) = 1$. This finishes the example.

Remark 8.2. By contrast with Example 3.1, (where we had two double cosets that could potentially contribute to $\mathrm{Hom}_{G^\theta}(\pi, 1)$, both cointributing one dimension whenever $\mathrm{Hom}_{G^\theta}(\pi, 1)$ was nonzero), in Example 8.1, we found that two K-G^θ double cosets could potentially contribute to $\mathrm{Hom}_{G^\theta}(\pi, 1)$, but after further analysis, we discovered that only one of the double cosets could actually give a nonzero contribution.

Remark 8.3. Example 8.1 is a special case of a more general family of examples discussed in Example 5.37 of [HM4].

9. CUSPIDAL G-DATA AND YU'S CONSTRUCTION

Yu's construction ([Y]) starts with a cuspidal G-datum Ψ. To each such Ψ, Yu associates an irreducible smooth representation $\kappa = \kappa(\Psi)$ of an open subgroup $K = K(\Psi)$ of G. The subgroup K contains the centre Z of G, and K/Z is compact. Yu proves that the representation $\pi = \pi(\Psi) = \mathrm{Ind}_K^G\, \kappa$ obtained by compact induction from κ is irreducible. Since π has matrix coefficients that are compactly supported modulo Z, π is supercuspidal.

We will write cuspidal G-data as 4-tuples: $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$. We will make a few remarks about the objects in these 4-tuples, as well as the associated K and κ. First $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \ldots, \mathbf{G}^d)$ is a tamely ramified twisted Levi sequence in $\mathbf{G}$. Here, d is a nonnegative integer, $\mathbf{G}^0 \subsetneq \cdots \subsetneq \mathbf{G}^d = \mathbf{G}$, each $\mathbf{G}^i$ is an F-group, and there exists a tamely ramified finite Galois extension E of F such that $\mathbf{G}^0$ splits over E, and $\mathbf{G}^i$, viewed as an E-group, is an E-Levi subgroup of $\mathbf{G}$. The groups $\mathbf{G}^i$ are called tamely ramified twisted Levi subgroups of $\mathbf{G}$. There is a cuspidality condition, namely that the centre Z^0 of $G^0 = \mathbf{G}^0(F)$ is compact modulo Z.

Example 9.1. Let ϖ be a prime element in F. Consider the centralizer $\mathbf{G}'$ of the matrix

$$\begin{pmatrix} 0 & \varpi & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varpi \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL_4(F).$$

Then

$$G' = \mathbf{G}'(F) = \left\{ g = \begin{pmatrix} a_1 & b_1\varpi & a_2 & b_2\varpi \\ b_1 & a_1 & b_2 & a_2 \\ a_3 & b_3\varpi & a_4 & b_4\varpi \\ b_3 & a_3 & b_4 & a_4 \end{pmatrix} \ \mid\ a_i,\, b_i \in F,\ \det g \neq 0 \right\} \simeq GL_2(F(\sqrt{\varpi})).$$

Note that $\mathbf{G}' = R_{F(\sqrt{\varpi})/F} GL_2$ splits over the tamely ramified extension $F(\sqrt{\varpi})$ of F, and $\mathbf{G}'(F(\sqrt{\varpi})) = GL_2(F(\sqrt{\varpi})) \times GL_2(F(\sqrt{\varpi}))$ is an $F(\sqrt{\varpi})$-Levi subgroup of $GL_4(F(\sqrt{\varpi}))$. The centre Z' of G' is isomorphic to $F(\sqrt{\varpi})^\times$, and $F(\sqrt{\varpi})^\times/F^\times$ is compact.

The element y belongs to $\mathcal{B}(G^0)$, and is such that $G^0_{y,0}$ is a maximal parahoric subgroup of G^0. Let $K^0 = N_{G^0}(G^0_{y,0})$ be the normalizer in G^0 of $G^0_{y,0}$. Next, ρ is an irreducible representation of K^0 such that $\rho\,|\,G^0_{y,0^+}$ is a multiple of the trivial representation of $G^0_{y,0^+}$, and $\rho\,|\,G^0_{y,0}$ contains a represntation that factors to an irreducible cuspidal representation of $G^0_{y,0}/G^0_{y,0^+}$. (Recalling the comments about depth zero supercuspidal representations in earlier sections, we see that these conditions imply that $\mathrm{Ind}_{K^0}^{G^0}\rho$ is an (irreducible) depth zero supercuspidal representation of G^0.)

Finally $\vec{\phi} = (\phi_0, \dots, \phi_d)$ is a sequence of quasicharacters, where ϕ_i is a quasicharacter of $G^i = \mathbf{G}^i(F)$, $0 \le i \le d$. Let r_i be the depth of ϕ_i (r_i is the smallest (nonnegative) real number such that $\phi_i\,|\,G^i_{y,r_i^+} \equiv 1$). Then we must have $0 < r_0 \cdots < r_{d-1}$ whenever $d > 0$. If $d > 0$ and ϕ_d is nontrivial, then $r_{d-1} < r_d$. If $d = 0$ and ϕ_d is nontrivial, then $0 < r_0 = r_d$. One of the most important conditions is that if $d > 0$ and $i \le d-1$, then ϕ_i be G^{i+1}-generic (relative to y). This is a regularity condition on ϕ_i (relative to G^{i+1}). The precise definition can be found in [Y].

Example 9.2. Let $G = GL_4(F)$ and let $\mathbf{G}'$ be as in Example 9.1. Set $\mathbf{G}^0 = \mathbf{G}'$ and $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G})$. Then $\vec{\mathbf{G}}$ is a tamely ramified twisted Levi sequence in $\mathbf{G}$. Let $y \in \mathcal{B}(G^0)$ be such that $G^0_{y,0} = GL_4(\mathfrak{o}_{F'})$, where $F' = F(\sqrt{\varpi})$, and let ρ be a representation of $F'^\times G^0_{y,0}$ such that $\rho\,|\,G^0_{y,0}$ factors to an irreducible cuspidal representation of $GL_2(\mathfrak{f}') = GL_2(\mathfrak{f})$. Let ϕ_0 be a quasicharacter of G^0 that has positive depth and is G-generic. For example, if ℓ is an integer greater than 1 and ψ is a character of F that is nontrivial on $\mathfrak{o}_F$ and trivial on $\mathfrak{p}_F$, then the map

$$k \mapsto \psi\left(\mathrm{trace}\left(\begin{pmatrix} 0 & \varpi^{-\ell+1} & 0 & 0 \\ \varpi^{-\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varpi^{-\ell+1} \\ 0 & 0 & \varpi^{-\ell} & 0 \end{pmatrix}(k-1)\right)\right)$$

from $G^0_{y,\ell-1/2} = 1 + \mathfrak{gl}_2(\mathfrak{p}_{F'}^{2\ell-1}) \subset 1 + \mathfrak{gl}_4(\mathfrak{p}_F)$ to $\mathbb{C}^\times$ defines a G-generic character of $G^0_{y,\ell-1/2}$. We could take ϕ_0 to be any quasicharacter of G^0 takes these values on the subgroup $G^0_{y,\ell-1/2}$. Let ϕ_1 be the trivialcharacter of G, and set $\vec{\phi} = (\phi_0, \phi_1)$. Then $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ is a cuspidal G-datum.

Example 9.3. Let $\varepsilon \in \mathfrak{o}_F^\times$ be a nonsquare. Set $\mathbf{G}^1 = \mathbf{G}'$, where $\mathbf{G}'$ is as in Example 9.1. Let $T = G^0$ be the centralizer of the matrix

$$A = \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

in $G^1 = GL_2(F') = GL_2(\mathbb{F}(\sqrt{\varpi}))$ (here, we continue to view G^1 as sitting inside $GL_4(F)$, as in Example 9.1). Then $T \simeq E^\times$, where $E = F(\sqrt{\varpi}, \sqrt{\varepsilon})$, and $\mathbf{G}^0 = \mathbf{T} = R_{E/F}GL_1$. Note that $E^\times/F^\times$ is compact. That is, $T/F^\times$ is compact. The sequence $\vec{\mathbf{G}} = (\mathbf{G}^0 = \mathbf{T}, \mathbf{G}^1, \mathbf{G})$ is a tamely ramified twisted Levi sequence in $\mathbf{G}$.

The group T has a unique maximal compact subgroup, namely $\mathfrak{o}_E^\times$. The normalizer of $\mathfrak{o}_E^\times$ in T is just T itself, so $K^0 = T$. Let ρ be the trivial character of T. Let ϕ_1 be the quasicharacter of G^1 that was labelled ϕ_0 in the previous example. Let ϕ_2 be the trivial character of G. We take a quasicharacter ϕ_0 of T that has the property

$$\phi_0(\gamma) = \psi(\mathrm{trace}(\varpi^{-1}A(\gamma-1))), \qquad \gamma \in 1 + \mathfrak{p}_E \subset GL_4(F).$$

With these choices $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ is a cuspidal G-datum.

Let $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ be a cuspidal G-datum. Then Yu defines a representation $\kappa = \kappa(\Psi)$ of an open, compact mod centre subgroup $K = K(\Psi)$ of G. He shows that $\pi = \pi(\Psi)$ is irreducible. As already mentioned, $K^0 = N_{G^0}(G^0_{y,0})$ is the normalizer of $G^0_{y,0}$ in G^0.

If $d = 0$, then $G = G^0$ and $\phi_d = \phi_0$ is a quasicharacter of G. In this case, $K = K^0$, $\kappa = \rho(\phi_0 \,|\, K^0)$, and $\pi = (\mathrm{Ind}_{K^0}^G \rho)\phi_0$ is a twist of a depth zero supercuspidal representation $\mathrm{Ind}_{K^0}^G \rho$ of G.

Suppose that $d > 0$. Let r_i be the depth of ϕ_i, $0 \leq i \leq d-1$. Let $s_i = r_i/2$. The group K^0 normalizes $G^i_{y,u}$ for all i and all $u \geq 0$. Also, if $j \geq i$ and $u \geq t$, then $G^i_{y,t}$ normalizes $G^u_{y,j}$. Consequently,

$$K = K(\Psi) := K^0 G^1_{y,s_0} G^2_{y,s_1} \cdots G^d_{y,s_{d-1}}$$

defines a subgroup of G. Define κ_{-1} to be equal to ρ on K^0, and then extend trivially across the subgroup $G^1_{y,s_0} \cdots G^d_{y,s_{d-1}}$ of K. Let $\kappa_d = \phi_d \,|\, K$. For $0 \leq i \leq d-1$, the restriction $\phi_i \,|\, K \cap G^i$ defines a character of $K \cap G^i$. We may use the genericity of ϕ_i relative to G^{i+1} to define a character $\hat{\phi}_i$ of G^{i+1}_{y,s_i^+} that agrees with ϕ_i on $G^i_{y,s_i^+} = (K \cap G^i) \cap G^{i+1}_{y,s_i^+}$. If $(K \cap G^i)G^{i+1}_{y,s_i^+} = (K \cap G^i)G^{i+1}_{y,s_i}$, then we may use ϕ_i and $\hat{\phi}_i$ to define a quasicharacter of $(K \cap G^i)G^{i+1}_{y,s_i}$ that extends to a quasicharacter κ_i of K. If $(K \cap G^i)G^{i+1}_{y,s_i^+} \neq (K \cap G^i)G^{i+1}_{y,s_i}$, then a Weil-Heisenberg construction is used to define an irreducible representation κ_i of K that has the property that $\kappa_i \,|\, K \cap G^i_{y,0+}$ is a multiple of ϕ_i and $\kappa_i \,|\, G^{i+1}_{y,s_i^+}$ is a multiple of $\hat{\phi}_i$. (For details, the reader may refer to [Y] and [HM4].) Finally, the representation κ is defined to be the (restriction to K of the) tensor product $\kappa_{-1} \otimes \kappa_0 \otimes \cdots \otimes \kappa_d$ of the representations $\kappa_{-1}, \kappa_0, \ldots, \kappa_d$.

10. GENERAL RESULTS DESCRIBING $\mathrm{Hom}_{G^\theta}(\pi, 1)$

Let $\mathbf{G}$ be a connected reductive F-group that splits over a tamely ramified extension of F. Let $G = \mathbf{G}(F)$. Let $\pi = \mathrm{Ind}_K^G \kappa$ be a tame supercuspidal representation of G, and let θ be an involution of G. As stated in Proposition 4.1,

$$\mathrm{Hom}_{G^\theta}(\pi, 1) \simeq \bigoplus_{g \in K\backslash G/G^\theta} \mathrm{Hom}_{K \cap G^{g\cdot\theta}}(\kappa, 1).$$

Next, results of [HM4] tell us that if $(g\cdot\theta)(K) \neq K$ for a particular $g \in G$, then $\mathrm{Hom}_{K\cap G^{g\cdot\theta}}(\kappa, 1) = 0$. Thus, when $\mathrm{Hom}_{G^\theta}(\pi, 1) \neq 0$, after replacing θ, κ and K by $g \cdot \theta$, $g^{-1}Kg$ and ${}^g\kappa$ if necessary, we can assume that $\theta(K) = K$ and $\mathrm{Hom}_{K^\theta}(\kappa, 1) \neq 0$. There exists a finite sequence $\{ g_1 = 1, g_2, \ldots, g_m \} \subset G$ with $Kg_iG^\theta \cap Kg_jG^\theta = \emptyset$ whenever $i \neq j$, $g_i\theta(g_i)^{-1} \in K$, and $(g \cdot \theta)(K) \neq K$ if $g \notin \cup_{i=1}^m Kg_iG^\theta$. Thus

$$\mathrm{Hom}_{G^\theta}(\pi, 1) \simeq \bigoplus_{i=1}^m \mathrm{Hom}_{K^{g_i\cdot\theta}}(\kappa, 1).$$

We will say more about the elements g_i shortly. If π has depth zero, then the results of [HM4] do not give further information about the terms $\mathrm{Hom}_{K^{g_i\cdot\theta}}(\kappa, 1)$. When the depth of π is positive, the main theorem of [HM4] (Theorem 5.26) relates the terms $\mathrm{Hom}_{K^{g_i\cdot\theta}}(\kappa, 1)$ to (quadratic) distinguished properties of the depth zero part ρ of a cuspidal G-datum Ψ such that $\pi = \pi(\Psi)$.

Theorem 10.1. *(*[HM4]*) Let Ψ be a cuspidal G-datum. Let $\pi = \pi(\Psi)$. We assume certain hypotheses concerning quasicharacters of the groups $G^i = \mathbf{G}^i(F)$, $0 \leq i \leq d$, where $\vec{\mathbf{G}} = (\mathbf{G}^0, \ldots, \mathbf{G}^d)$. Suppose that* $\mathrm{Hom}_{G^\theta}(\pi, 1) \neq 0$. *Then the cuspidal G-datum $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ can be chosen to satisfy:*

1. *$\theta(G^i) = G^i$, $0 \leq i \leq d-1$, and $\theta(G^0_{y,0}) = G_{y,0^0}$ (which imply $\theta(K^0) = K^0$ and $\theta(K) = K$).*
2. *$\phi_i \circ \theta = \phi_i^{-1}$, $0 \leq i \leq d$.*
3. *Let $\phi = \prod_{i=0} \phi_i \,|\, G^0$. There exists a character χ of K^θ such that $\chi^2 \equiv 1$ and*

$$\mathrm{Hom}_{K^\theta}(\kappa, 1) \simeq \mathrm{Hom}_{(K^0)^\theta}(\rho, \phi\chi).$$

Above, $\mathrm{Hom}_{(K^0)^\theta}(\rho, \phi\chi)$ is the set of linear functionals λ on the space of W of ρ that satisfy $\lambda(\rho(k)w) = (\phi\chi)(k)\lambda(w)$ for every $k \in (K^0)^\theta$. (We remark that the square of $\chi(\phi \,|\, (K^0)^\theta)$ is trivial.)

Let $\tau : G \to G$ be defined by $\tau(g) = g\theta(g)^{-1}$. Consider the set $K^0 \cap \tau(G)$. This set is a union of finitely many K^0-orbits, for the action of K^0 defined by $k \cdot h = kh\theta(k)^{-1}$, $k \in K^0$, $h \in K^0 \cap \tau(G)$. Choose $\{ h_1 = 1, h_2, \ldots, h_m \}$ to be representatives for these K^0-orbits. Fix $g_i \in G$ such that $g_i\theta(g_i)^{-1} = h_i$, $1 \leq i \leq m$. Then, as shown in [HM4] (although the statement in [HM4] is not identical to the statement here), writing $\theta_i = g_i \cdot \theta_i$, we have

Theorem 10.2. *Assuming that Ψ is as in the above theorem,* $\mathrm{Hom}_{K^{g\cdot\theta}}(\kappa, 1) = 0$ *whenever $\tau(g) \notin K$. Furthermore, there exists a character χ_i of $(K^0)^{\theta_i}$ whose square is trivial and* $\mathrm{Hom}_{K^{\theta_i}}(\kappa, 1) \simeq \mathrm{Hom}_{(K^0)^{\theta_i}}(\rho, \phi\chi_i)$*, $1 \leq i \leq m$. Thus*

$$\mathrm{Hom}_{G^\theta}(\pi, 1) \simeq \bigoplus_{i=1}^m \mathrm{Hom}_{(K^0)^{\theta_i}}(\rho, \phi\chi_i).$$

Remark 10.3. Analyzing $\mathrm{Hom}_{(K^0)^{g_i\cdot\theta}}(\rho, \phi\chi_i)$ is similar to analyzing $\mathrm{Hom}_{(K^0)^{g_i\cdot\theta}}(\rho, 1)$. Indeed, in some cases, $(\phi \,|\, (K^0)^{\theta_i})\chi_i$ can be trivial.

In the general setting, it can be difficult to determine $\mathrm{Hom}_{G^\theta}(\pi, 1)$, due to difficulty parametrizing the double cosets Kg_iG^θ, computing the χ_i's, and computing the dimensions of the spaces $\mathrm{Hom}_{K^{g_i}\cdot\theta}(\kappa, 1)$.

In certain special cases, things simplify. We say that π if *toral* if for every (equivalently, some) cuspidal G-datum Ψ such that $\pi \simeq \pi(\Psi)$, the group $\mathbf{G}^0$ is a torus. In that case, choose a Ψ as in the theorem, and set $\mathbf{T} = \mathbf{G}^0$, $T = \mathbf{T}(F)$. According to the theorem, we have $\theta(G^i) = G^i$ for and $\phi_i \circ \theta = \phi_i^{-1}$ for $0 \leq i \leq d-1$. Note that $K^0 = T = G^0$, so ρ is a depth zero quasicharacter of T. We may adjust Ψ and take ρ to be trivial as long as we replace ϕ_0 by $\phi_0\rho$ and replace $\vec{\phi}$ by $(\phi_0\rho, \phi_1, \dots, \phi_d)$. However if we do this, we might have $(\phi_0\rho) \circ \theta \neq (\phi_0\rho)^{-1}$. Lets assume that we have made this adjustment. Then we know that $\phi_0 \circ \theta \,|\, T_{0+} = \phi_0^{-1} \,|\, G_{0+}$, but this equality might not extend to all of T. Results of [HM4] show that $\chi_i = \chi_1 = \chi$ for $1 \leq i \leq m$ (in the toral case). Note that we have $h_i = g_i\theta(g_i)^{-1} \in T = G^0$, and T is abelian. Thus $\theta_i \,|\, T = \theta \,|\, T$.

Toral case: more to be added here....

11. Equivalence of tame supercuspidal representations

12. Distinguished tame supercuspidal representations of general linear groups

More to be added .....

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