MAT 247S - Problem Set 5

Due Thursday March 12

Questions 3, 4, 5, 9b), 9c) and 10b) will be marked.

1. Let V be a finite-dimensional vector space over a field F and let $T \in \mathcal{L}(V)$. Let x be a nonzero vector in V and let W be the T-cyclic subspace generated by x.
   a) Let $y \in V$. Prove that $y \in W$ if and only if there exists a polynomial $g(t) \in P(F)$ such that $y = g(T)(x)$.
   b) Prove that the polynomial $g(t)$ of part a) can always be chosen so that its degree is less than or equal to $\dim(W)$.

2. Let T be a linear operator on a finite-dimensional vector space V over a field F, and suppose that V is a T-cyclic subspace of itself. Let U be a linear operator on V. Prove that $UT = TU$ if and only if $U = g(T)$ for some polynomial $g(t) \in P(F)$. (*Hint*: Let $x \in V$ be a vector such that the T-cyclic subspace generated by x is equal to V. Let $y = U(x)$. Choose $g(t)$ as in question 1.)

3. Let V be a finite-dimensional vector space over a field F. Suppose that $T \in \mathcal{L}(V)$ has the property that the characteristic polynomal of T splits over F. Prove that any nonzero T-invariant subspace of V contains an eigenvector of T.

4. Let V be an n-dimensional vector space over a field F. Suppose that $T \in \mathcal{L}(V)$. Let d be the degree of the minimal polynomial of T. Let $x \in V$ and let W be the T-cyclic subspace of V generated by x. Prove that $\dim(W) \leq d$.

5. Let V be an n-dimensional vector space over a field F. Suppose that $T \in \mathcal{L}(V)$ and λ_1, λ_2 and $\lambda_3 \in F$ are distinct eigenvalues of T. Let x_j be an eigenvector of T corresponding to the eigenvalue λ_j, $1 \leq j \leq 3$. Let $y_1 = x_1$, $y_2 = x_1 + x_2$ and $y_3 = x_1 + x_2 + x_3$. Let W_j be the T-cyclic subspace of V generated by y_j. Prove that $\dim W_j = j$.

6. Let V be a finite-dimensional vector space over a field F. Let $T \in \mathcal{L}(V)$ and let $W = \mathrm{span}\{ I_V, T, T^2, T^3, \ldots, T^j, \ldots \} \subset \mathcal{L}(V)$. Prove that $\dim(W)$ is equal to the degree of the minimal polynomial of T.

7. Let T be a linear operator on a finite-dimensional complex inner product space. Let $p(t) = t^d + c_{d-1}t^{d-1} + \cdots + c_1 t + c_0$ be the minimal polynomial of T. Let $\bar{p}(t) = t^d + \bar{c}_{d-1}t^{d-1} + \cdots + \bar{c}_1 t + \bar{c}_0$. (Here, $\bar{c}_j$ denotes the complex conjugate of c_j, $0 \leq j \leq d-1$.) Let T^* be the adjoint of T.
   a) Prove that $\bar{p}(T^*) = 0$.
   b) Prove that $\bar{p}(t)$ is the minimal polynomial of T^*.

Notation used below: If $f(t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0$, and $a_0, \ldots, a_{d-1}$ are complex numbers, then

$$\begin{aligned} & f(-t) = a_d(-1)^d t^d + a_{d-1}(-1)^{d-1}t^{d-1} + \cdots - a_1 t + a_0 \\ \text{and} \quad & f(t^2) = a_d t^{2d} + a_{d-1}t^{2d-2} + \cdots + a_1 t^2 + a_0. \end{aligned}$$

8. Let $V = P(\mathbb{R})$ and define $T \in \mathcal{L}(V)$ by $T(f) = f'$, where f' denotes the first derivative of f. Prove that if $g(t) \in P(\mathbb{R})$ is nonzero, then $g(T) \neq T_0$. (Therefore T has no minimal polynomial.)

9. Let $T : V \to V$ be a linear operator on a finite-dimensional complex vector space V.

   a) Prove that if T is diagonalizable then T^2 is diagonalizable.

   For parts b) and c), let $p(t)$ be the minimal polynomial of T and let $g(t)$ be the minimal polynomial of T^2.

   b) Prove that $p(t)$ divides $g(t^2)$. (*Do not assume that T is diagonalizable.*)

   c) Suppose that T is diagonalizable, T is invertible, and $p(t) = p(-t)$. Prove that $p(t) = g(t^2)$.

10. Let T be a linear operator on a finite-dimensional vector space V over a field F. Let W_1 and W_2 be nonzero T-invariant subspaces of V such that $V = W_1 \oplus W_2$. (That is, $W_1 \cap W_2 = \{\mathbf{0}\}$ and $V = W_1 + W_2$. Equivalently, every vector in V can be expressed uniquely as the sum of a vector in W_1 and a vector in W_2.) Let $p_1(t)$ and $p_2(t)$ be the minimal polynomials of the restrictions T_{W_1} and T_{W_2} of T to W_1 and W_2, respectively.

    a) Prove that $p_1(t)$ and $p_2(t)$ divide the minimal polynomial of T.

    b) Find an example of a V, W_1, W_2 and T as above such that $p_1(t)p_2(t)$ is not equal to the minimal polynomial of T.

    c) Find an example of a V, W_1, W_2 and T as above such that $p_1(t)p_2(t)$ is equal to the minimal polynomial of T.

11. §7.3, #16.