MAT247S, 2009 Winter, Problem Set 6 Solution

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1.(a) Since $W = \bigoplus_{i=1}^{l} F((T-\lambda I)^{i-1}(x))$ is $(T-\lambda I)$-invariant and λI-invariant, it is also T-invariant.
(b) By definition $(T-\lambda I)^l(x) = 0$, and $(-1)^l(t-\lambda)^l$ is of deg$= l = \dim(W)$.
(c) If the minimal polynomial is $(t-\lambda)^k$ for some $k < l$, then $(T-\lambda I)^k(x) = 0$. Since l is the smallest one giving $(T-\lambda I)^l(x) = 0$, we must have $k = l$.

2. By definition of initial vectors, for $i = 1, 2$ let $x_i = (T-\lambda I)^{m_i-1}(y_i)$ with $(T-\lambda I)^{m_i}(y_i) = 0$. If $(T-\lambda I)^{l_1}(y_1) = (T-\lambda I)^{l_2}(y_2)$ for some l_i, then one can show $m_1 - l_1 = m_2 - l_2$. From this one can show $x_1 = (T-\lambda I)^{m_1}(y_1) = (T-\lambda I)^{m_2}(y_2) = x_2$.

3.(a) Since K_λ is T-invariant, so it is $g(T)$-invariant.
(b) (8 marks total) ($\Rightarrow$) If $g(\lambda) = 0$, then
(2 marks) pick a non-zero eigenvector $x \in K_\lambda$ (such eigenvector must exist),
(1 mark) we have $U(x) = g(T)(x) = g(\lambda)x = 0$, so U is not invertible.
($\Leftarrow$) Suppose U is not invertible, so
(1 mark) there is non-zero vector $x \in K_\lambda$ that $g(T)(x) = 0$.
(1 mark) Since $x \in K_\lambda$, there is positive integer p so that $(T-\lambda I)^p(x) = 0$.
(2 marks) If we choose the smallest such p, then $y = (T-\lambda I)^{p-1}(x)$ is non-zero, and $(T-\lambda I)(y) = 0$, i.e. y is a non-zero λ-eigenvector.
(1 mark) We have $U(y) = g(T)(T-\lambda I)^{p-1}(x) = (T-\lambda I)^{p-1}g(T)(x) = (T-\lambda I)^{p-1}(0) = 0$. On the other hand $U(y) = g(T)(y) = g(\lambda)y \neq 0$. We derive a contradiction. Therefore U is invertible.

REMARK In ($\Leftarrow$), a number of students pick a λ-eigenvector x, then by saying $g(T)(x) = g(\lambda)x \neq 0$ and conclude $\ker(g(T)) = 0$ But what you have shown is just $\ker(g(T)) \cap \ker(T-\lambda I) \neq 0$.

Alternative solution: Consider $S = T_{K_\lambda}$ first. We know there is a basis for K_λ so that S can be block-diagonalized so that each block is of the Jordan-form $\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$. Under such basis $U = g(S) = \begin{pmatrix} g(\lambda) & * & * & * & * \\ & & \ddots & * & * \\ 0 & 0 & 0 & g(\lambda) & * \\ 0 & 0 & 0 & 0 & g(\lambda) \end{pmatrix}$ on each block. (Here * is not necessary 0 or 1, but the value is not important here.) Hence U is upper triangular with all diagonal entries being $g(\lambda)$. It is clear that U is invertible iff $g(\lambda) \neq 0$.

4.(a) it is because each K_{λ_j} are linearly independent.

(b) $\supseteq$ is from definition of K_{λ_j}, and $\subseteq$ is from the fact (iii) above.

(c) (6 marks total) Since $p(T_{K_{\lambda_j}}) = p(T)_{K\lambda_j} = 0$, we have (2 marks) the minimal polynomial $m_{T_{K_{\lambda_j}}}$ divides p.
(2 marks) If $t - \lambda_i$ divides $m_{T_{K_{\lambda_j}}}$ for $\lambda_i \neq \lambda_j$, then $T - \lambda_i I$ is not invertible on K_{λ_j}. this contradicts the fact (iii).
(2 marks) So $m_{T_{K_{\lambda_j}}}$ must of the form $(t - \lambda_j)^k$ for some $k < \lambda_j$, then for polynomial $p'(t) = (t - \lambda_1)^{l_1} \cdots (t - \lambda_j)^{l_j - 1} \cdots (t - \lambda_k)^{l_k}$ it is easy to show $p'(T) \equiv 0$ on each K_{λ_i}, hence is T_0 by fact (i) given. But $\deg(p') < \deg(p)$ contradicts the minimality of p. So $k = l_j$.

(d) (6 marks total) Let the cycle be $\gamma = \{x, \ldots, (T - \lambda_j I)^k(x)\}$ and $W = \mathrm{span}(\gamma) \subseteq K_{\lambda_j}$.
(3 marks) Since the minimal polynomial of T_W is $(t - \lambda_j)^k$, we must have $k \leq l_j$.
(3 marks) If all such cycle have length strictly less than l_j, then $(T - \lambda_j I)^{l_j - 1}$ kills K_{λ_j}, contradicting the minimality of p. Hence at least one cycle has length λ_j.

5.(a) Choose a basis and take T to be block-diagonalized, each block is the Jordan form $\begin{pmatrix} c & 1 & 0 & \ldots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & c & 1 \\ 0 & 0 & 0 & 0 & c \end{pmatrix}$, with maximum size d.
(b) The characteristic polynomial must be of the form $(-1)^n(t-c)^{n-1}(t-d)$. The minimality of p forcing $d = c$.
(c) By (b) under suitable bases $[T_1]_{\beta_1} = [T_2]_{\beta_2} = \begin{pmatrix} c & 1 & 0 & \ldots & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}$. Take U to be the matrix of changing basis.
(d) $T_1 = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}$, $T_2 = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$.

6. (Total 25 marks)
(a) (3 marks) $ch_T(t) = (t - i)^2(t + i)^2$ (2 marks) eigenvalues are $i, -i$.
(b) (2 marks) $\mathrm{rank}(T - iI) = 3$ (2 marks) $\mathrm{rank}(T + iI) = 2$
(1 mark) for calculation
(c) (2 marks) $m_T(t) = (t - i)^2(t + i)$
(3 marks) Since by part (b) $T + iI$ kills K_{-i}, while $T - iI$ does not kill K_i but $(T - iI)^2$ does.
(d) (4 marks) $\begin{pmatrix} i & 1 & & \\ & i & & \\ & & -i & \\ & & & -i \end{pmatrix}$, which can be directly read from the minimal polynomial, or use dot diagram.
(e) (4 marks) $K_i = \mathrm{span}\{v = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, (T - iI)v = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}\}$.
(2 marks) for calculation

REMARK One compute $K_{-i} = \text{span}\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$, with each vector gives cycle of length 1.

7. Jordan basis $\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$, with Jordan form $\begin{pmatrix} 2 & 1 & & \\ & 2 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$

8. One compute $U = T^2 + T - I = \begin{pmatrix} 5 & 5 & 1 \\ & 5 & 5 \\ & & 5 \end{pmatrix}$. Using Young diagram (or whatever method) one show the Jordan form of U should be $\begin{pmatrix} 5 & 1 & \\ & 5 & 1 \\ & & 5 \end{pmatrix}$.