MAT247S, 2009 Winter, Problem Set 7 Solution

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1. Recall T is diagonalizable if and only if $m_T(t)$ has no factor of multiplicity greater than 1. Now $m_T(t)$ divides $t^{r+2} - 4t^r = t^r(t-2)(t+2)$. Hence indeed $m_T(t)$ divides $t(t-2)(t+2)$ and $T^3 - 4T = T_0$. Conversely if $T^3 - 4T = T_0$ then $m_T(t)$ divides $t(t-2)(t+2)$. It cannot have factor with multiplicity greater than 1 and so T is diagonalizable.

2.(a) Take basis $\{1, t, t^2\}$ for $V = P_2(\mathbb{R})$ and write the vectors in β as column form, e.g. $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Then write the matrix whose i-th column being v_i. The matrix is $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Its inverse is $\begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$. The dual basis has coefficients being the i-th row of the inverse matrix, so $f_1(a + bt + ct^2) = 1/2a + 1/2b + 1/2c$, etc.
(b) Under the basis above one can show $f(a + bt + ct^2) = a + 3b + 4c$. Solving $\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 \end{pmatrix}$ one get $\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \end{pmatrix}$. So $f = 4f_1 + 2f_2 + 3f_3$.

3.(a)(b) (9 marks total)
(3 marks) Write f_i in row vector, e,g, $f_1 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$, and form the matrix whose i-th row is the row vector of f_i. The matrix is $\begin{pmatrix} -1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$.
(3 marks) The inverse of the matrix is $\begin{pmatrix} -1/3 & 1/3 & 0 \\ 2/3 & 1/3 & 0 \\ -2/3 & -1/3 & -1 \end{pmatrix}$
(3 marks) The i-th column of the inverse gives the coefficients of p_i, so $p_1 = -1/3 + 2/3t - 2/3t^2$, etc.

4(a) (5 marks total)
(1 mark) Since $\dim(V) = n = \#\{y_i\}$, it suffices to show $\{y_i\}$ are linearly independent. If $\sum_i a_i y_i = 0$, then $\langle \sum_i a_i y_i, x \rangle = 0$ for all $x \in V$.
(2 marks) Write $x = \sum_j b_j x_j$, then $\langle \sum_i a_i y_i, x \rangle = \sum_i a_i b_i = 0$ for any b_i.
(2 marks) Choosing $b_i = 1$ and $b_j = 0$ for $j \neq i$, we have $a_i = 0$.

(b) (5 marks total)
(2 marks) If $\beta = \beta'$ then $\langle x_i, x_j \rangle = \langle x_i, y_j \rangle = f_j(x_i) = \delta_{ij}$. This means β is orthogonal.
(1 mark) Conversely, if β orthogonal, then $\langle x_i, x_j \rangle = \delta_{ij} = f_j(x_i) = \langle x_i, y_j \rangle$. So $\langle x_i, x_j - y_j \rangle = 0$ for all x_i.
(2 marks) Since $\{x_i\}$ is a basis for V, by taking linear combination we have $\langle x, x_j - y_j \rangle = 0$ for all $x \in V$. Hence $x_j = y_j$, and $\beta = \beta'$.

5. One compute $T^t(f_x)(y) = f_x(T(y)) = \langle T(y), x\rangle$ which equals $f_{T^*(x)}(y) = \langle y, T^*(x)\rangle$. The above is true for all $x, y \in V$, so T^* is uniquely determined.

6. One compute $f_1(a+bt) = a + 1/2b$ and $f_2(a+bt) = 2a+2b$, and $\begin{pmatrix} 1 & 1/2 \\ 2 & 2 \end{pmatrix}$ is invertible.

7. One compute $T(A+Bt) = (-A-2B, A+B)$, and (a) $T^t f(A+Bt) = -3A-4B$.
(b)(c) The transpose is $T^t = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$.

8.(b) (8 marks total) Refer to Prof Murnaghan's Notes. To check H is a subgroup,
(1 mark) for checking $e = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \in H$, or at least showing $H \neq \emptyset$.
(2 marks) for checking if $a, b \in H$, then $ab^{-1} \in H$.
(2 marks) for showing H is abelian.
(3 marks) for showing H is not normal.

10.(a) (8 marks total) Refer to Prof Murnaghan's Notes.
(2 marks) for showing for each i, the elements $\{x_i x_j\}_{j=1}^n$ are all distinct.
(3 marks) for showing $e \in H$.
(3 marks) for showing $x_i^{-1} \in H$ for all i.

15.(b) (10 marks total) Refer to Prof Murnaghan's Notes.
(3 marks) for showing the map is a homomorphism.
(4 marks) for identifying the kernel.
(3 marks) for identifying the image.