MAT 247H1S - ALGEBRA II

Review questions and term test info

Term test: Tuesday February 24th, 1:10-3pm HA403

No aids allowed.

Test material - course material up to and including the spectral theorem - that is, material covered in the first four problem sets.

Relevant sections of text: 6.1, 6.2, 6.3 (excluding least squares approximations and minimal systems to systems of equations); 6.4, 6.5(excluding rigid motions, orthogonal operators on \mathbb{R}^2 , conic sections), 6.6 (up to Theorem 6.5); see also the notes on orthogonal projections posted on the course home page.

The test will consist of approximately 5 questions; at least two of which will be concrete computational questions, and at least two of which will be proof questions.

Questions similar to (in some cases identical to) the following problem set questions have previously appeared on term tests or final exams:

Problem Set 1: 1c), 1d), 3, 4, 5, 7c), 9 Problem Set 2: 1a), 1b), 3, 5, 6, 8, 9, 12 Problem Set 3: 1, 2, 4, 6, 9, 10 Problem Set 4: 1a), 2a), 2b), 3a), 3b), 4, 6

Additional review questions

1. Suppose that V is a vector space over $F = \mathbb{R}$ or $F = \mathbb{C}$. Let $\langle \cdot, \cdot \rangle : V \times V \to F$ be an inner product on V. Let $T: V \to V$ be a linear operator. Suppose that the function $\langle \cdot, \cdot \rangle' : V \times V \to F$ defined by

$$\langle x, y \rangle' = \langle T(x), y \rangle, \qquad x, y \in V$$

is another inner product on V. Prove that T is self-adjoint and every eigenvalue of T is a positive real number.

2. Let $V = P_2(\mathbb{R})$. Define

$$\langle f_1, f_2 \rangle = 3(f_1(0) + f_1(1))f_2(0) + 4f_1(-1)f_2(-1), \qquad f_1, f_2 \in P_2(\mathbb{R}).$$

Show that \langle , \rangle is not an inner product on V. (Demonstrate that at least one property of inner product is not satisfied.)

3. Let $V = \mathbb{R}^3$, with the inner product

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = 2 a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2 + a_3 b_3, \qquad a_j, b_j \in \mathbb{R}, 1 \le j \le 3.$$

Define $T \in \mathcal{L}(V)$ by

$$T(a_1, a_2, a_3) = ((a_1 + a_3)/2, a_2, (a_1 + a_3)/2), \quad a_j \in \mathbb{R}, 1 \le j \le 3.$$

- a) Find an orthonormal basis for R(T).
- b) Show that T(x) = x for all $x \in R(T)$.
- c) Is T equal to the orthogonal projection of V on the subspace W = R(T) of V? Justify your answer.
- 4. For each inner product space below, find an example of a $T \in \mathcal{L}(V)$ having the specified properties:

- a) Let $V = \mathbb{R}^3$ with the standard inner product. Required properties: T(0,1,0) = (4,0,-1)and $T^* = -T$. Give the value T(a,b,c) for all $a, b, c \in \mathbb{R}$ and verify that T satisfies the required properties.
- b) Let $V = \mathbb{C}^2$ with the standard inner product. Required properties: T(1, 1) = (i, -i) and T is unitary. Give the value T(a, b) for all a and $b \in \mathbb{C}$ and verify that T satisfies the required properties.
- 5. Suppose that $A \in M_{n \times n}(\mathbb{R})$ is symmetric, that is $A^t = A$. Prove that there exists a symmetric matrix $B \in M_{n \times n}(\mathbb{R})$ such that $B^3 = A$.
- 6. Let V be a finite-dimensional inner product space. Let $T_1 \in \mathcal{L}(V)$ and let $T = T_1 T_1^*$.
 - a) Prove that if λ is an eigenvalue of T, then λ is a real number and $\lambda \ge 0$.
 - b) Prove that there exists $T_2 \in \mathcal{L}(V)$ such that T_2 is self-adjoint and $T = T_2^2$.
- 7. Suppose that V is a finite-dimensional complex inner product space and $T: V \to V$ is a linear operator having the property that $T^* = iT$.
 - a) Prove that every eigenvalue of T has the form t(1-i), with t a real number.
 - b) Show that $I_V + T^2$ and $I_V T^2$ are normal.
 - c) Show that $I_V + T^2$ and $I_V T^2$ are invertible.
 - d) Let $U = (I_V + T^2)(I_V T^2)^{-1}$. Prove that U is unitary.
- 8. Let W be a nonzero subspace of a finite-dimensional inner product space V. Prove that the following are equivalent:
 - (i) There exists a $T \in \mathcal{L}(V)$ such that T is orthogonal and $T(W) \subset W^{\perp}$.
 - (ii) $\dim W \le (\dim V)/2$.
- 9. Let $V = \mathbb{C}^4$ with the standard inner product. Find an example of a $T \in \mathcal{L}(V)$ such that T is unitary and

 $\{T(1,0,i,0), T(0,1,0,i)\} \subset \text{Span}\{(1,0,-i,0), (0,1,0,-i)\}.$

Please explain why the above inclusion holds for your T, and explain why T is unitary.