CHAPTER 3

Representations of $SL_2(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field of order q. Then there exists a prime p and a positive integer ℓ such that $q = p^\ell$. For convenience, we assume that p is odd. We also assume that $q \neq 3$. Let $G = SL_2(\mathbb{F}_q)$ be the group of 2×2 matrices with entries in $\mathbb{F}_q$ and determinant equal to 1. In this chapter, we construct the (characters of the) irreducible representations of G.

Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times,\, b \in \mathbb{F}_q \right\}$. Let A be the group of diagonal matrices in G, and let $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_q \right\}$. Note that $B = A \ltimes N \simeq \mathbb{F}_q^\times \ltimes \mathbb{F}_q$. Hence $|B| = q(q-1)$.

Let $g \in G$. Note that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin B$ if and only if $c \neq 0$. In that case, $b = c^{-1}(ad - 1)$. Because a and d may be chosen freely in $\mathbb{F}_q$ and c may be chosen freely in $\mathbb{F}_q^\times$, it follows that the number of elements in the complement of B in G is equal to $q^2(q-1)$. Hence $|G| = q(q-1) + q^2(q-1) = q(q^2-1)$.

Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The double coset BwB is a disjoint union of right B cosets. Let b_1, $b_2 \in B$. Then $Bwb_1 = Bwb_2$ if and only if $b_1 b_2^{-1} \in w^{-1}Bw \cap B$. Note that

$$w^{-1} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} w = \begin{pmatrix} a^{-1} & 0 \\ -b & a \end{pmatrix}. \tag{3.1}$$

It follows that $wBw^{-1} \cap B = A$, and as g ranges over a set of (right) coset representatives for $A\backslash B$, then g ranges over a set of representatives for the right B cosets in BwB. Note that $A\backslash B \simeq N$. Hence we may (and do) view N as a set of coset representatives for $A\backslash B$ and for the right B cosets in BwB. Now $|N| = q$. Thus BwB contains exactly q right B cosets. Hence $|BwB| = q|B| = q^2(q-1)$. The subset $B \amalg BwB$ contains $q(q-1) + q^2(q-1) = q(q^2-1)$ elements, so must equal G.

Lemma. *G is generated by B and w, and there are two B-B double cosets in G, namely B and BwB. Also*

$$G = \amalg_{x \in \mathbb{F}_q} Bw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \amalg B.$$

We will begin to study the representations of G by looking at representations of G which are induced from linear characters (one-dimensional representations) of B. Note that the derived group (commutator subgroup) of B is equal to N. Hence there is a bijection between the set of linear characters of $A \simeq B/N$ and the set of linear characters of B. Now $A \simeq \mathbb{F}_q^\times$ and $\mathbb{F}_q^\times$ is a cyclic group of order $q-1$. Let $\zeta \in \mathbb{C}$ be a primitive root of unity of order $q-1$, and let α be a generator of $\mathbb{F}_q^\times$. For each m such that $0 \leq m \leq q-2$, the map $\alpha^j \mapsto \zeta^{jm}$ defines a linear character of $\mathbb{F}_q^\times$. It is clear that these characters are distinct. Since there are $q-1$ of them, this gives a complete list of the linear characters of $\mathbb{F}_q^\times$. Note that there are two characters of $\mathbb{F}_q^\times$ whose squares are trivial. One is the trivial representation of $\mathbb{F}_q^\times$, corresponding to $m = 0$, and the other one, corresponds to $m = (q-1)/2$, and takes the value -1 on non-squares in $\mathbb{F}_q^\times$ and 1 on squares.

Let τ_0 be a character of $\mathbb{F}_q^\times$. The associated character τ of B is defined by $\tau\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \tau_0(a)$. Looking at equation (3.1), we see that the character τ^w of $B^w = wBw^{-1} \cap B = A$ (notation as in Chapter 2) is given by

(3.2)

$$\tau^w \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \tau\left(w^{-1} \begin{pmatrix} a & 0 \\ 0 & a^{-1}\, w^{-1} \end{pmatrix} w \right) = \tau \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \tau_0(a)^{-1}, \qquad a \in \mathbb{F}_q^\times.$$

Hence

$$\mathrm{Hom}_{B^w}(\tau^w, r_B^{B^w}\tau) = \mathrm{Hom}_A(\tau_0^{-1}, \tau_0) = \begin{cases} \mathbb{C}, & \text{if } \tau_0^2 = 1 \\ 0, & \text{if } \tau_0^2 \neq 1 \end{cases}$$

Combining this with results from Chapter 2 concerning induced representations, we have

Lemma. *$i_B^G\tau$ is irreducible if and only if $\tau_0^2 \neq 1$. If $\tau_0^2 = 1$, then* $\dim \mathrm{Hom}_G(i_B^G\tau, i_B^G\tau) = 2$.

Given $a \in \mathbb{F}_q^\times$, set $s_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. It is easy to check that if $a \neq \pm 1$, then A is the centralizer of s_a in G. Hence if $a \neq \pm 1$, the order $|\mathrm{cl}(s_a)|$ of the conjugacy class $\mathrm{cl}(s_a)$ in G is equal to $|G|/|A| = q(q+1)$. Note that if a, $b \in \mathbb{F}_q$, then $gs_ag^{-1} = s_b$ if and only if s_a and s_b have the same eigenvalues. If $a \neq \pm 1$, this is equivalent to $b \in \{\, s_a, s_{a^{-1}} \,\}$. Thus, when $a \neq \pm 1$, $\mathrm{cl}(s_a) = \mathrm{cl}(s_b)$ if and only if $b = a$ or $b = a^{-1}$. Now the centre of G is equal to $\{\, s_1, s_{-1} \,\}$, so there are two single-element conjugacy classes in G. According to the above there are $(q-3)/2$ non-central conjugacy classes which contain elements of A, each such class containing $q(q+1)$ elements. If $a \in \mathbb{F}_q^\times$ and $a \neq \pm 1$, then it is easy to check that $\mathrm{cl}_B(s_a) = \{\, gs_ag^{-1} \mid g \in N \,\} = s_aN$. We can now conclude that

$$\mathrm{cl}_G(s_a) \cap B = s_aN \amalg s_{a^{-1}}N, \qquad a \in \mathbb{F}_q^\times,\ a \neq \pm 1.$$

Note that $\mathrm{cl}_G(\pm I) = \pm I = \mathrm{cl}_G(\pm I) \cap B$. Applying the Frobenius Character Formula, we can compute $\chi_{i_B^G\tau}$ on A.

Lemma. *Let $a \in \mathbb{F}_q^\times$. Then*

$$\chi_{i_B^G\tau}(s_a) = |G||B|^{-1}|\mathrm{cl}_G(s_a)|^{-1}(|cl_B(s_a)|\tau(s_a) + |\mathrm{cl}_B(s_{a^{-1}})|\tau(s_{a^{-1}})) = \tau_0(a) + \tau_0(a^{-1}),\ a \neq \pm 1.$$

and $\chi_{i_B^G\tau}(s_{\pm 1}) = (q+1)\tau_0(\pm 1)$.

Let ψ be a nontrivial character of (the additive group) $\mathbb{F}_q$. Let $t \in \mathbb{F}_q$. It is easy to see that the function $x \mapsto \psi_t(x) := \psi(tx)$ also defines a character of $\mathbb{F}_q$, and $\psi_s = \psi_t$ if and only if $s = t$. Hence $\{ \psi_t \mid t \in \mathbb{F}_q \}$ is the set of characters of $\mathbb{F}_q$. We can also view this set as the set of characters of N, since $N \simeq \mathbb{F}_q$.

Let $t \in \mathbb{F}_q$. Define a function $f_t^\tau : G \to \mathbb{C}$ by $f_t^\tau(b) = 0$ for $b \in B$ and

$$f_t^\tau \left(bw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \tau(b)\psi_t(x), \qquad b \in B,\ x \in \mathbb{F}_q.$$

Clearly $f_t^\tau \neq 0$ and f_t^τ belongs to the space of $i_B^G \tau$. Let $\mathcal{V}_\tau^o$ be the subspace of the space $\mathcal{V}_\tau$ of $i_B^G \tau$ which consists of those functions which are supported in BwB. Then the set $\{ f_t^\tau \mid t \in \mathbb{F}_q \}$ is a basis of $\mathcal{V}_\tau^o$. Note that

$$\begin{aligned}\left(i_B^G \tau \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} f_t^\tau \right) \left(bw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) &= f_t^\tau \left(bw \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \right) \\ &= \tau(b)\psi_t(x+y) = \psi_t(y) f_t^\tau \left(bw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right).\end{aligned}$$

Therefore (for t fixed) the span of f_t^τ is N-invariant and the restriction of $r_G^N(i_B^G \tau)$ to this one-dimensional space equivalent to the character ψ_t of N. It follows that the restriction of $r_G^N(i_B^G \tau)$ to V_τ^o is equivalent to the regular representation of N, since it is the direct sum of all of the characters of N, each occurring exactly once.

Let f_B^τ be the function which is zero on BwB and satisfies $f_B^\tau(b) = \tau(b)$, $b \in B$. The space $\mathcal{V}_\tau'$ of $\mathcal{V}_\tau$ consisting of functions supported on B is one-dimensional, and is spanned by the function f_B^τ. It is clear that the restriction of $r_G^N(i_B^G \tau)$ to $\mathcal{V}_\tau'$ is equivalent to the trivial representation of N.

Lemma.
(1) $r_G^N(i_B^G \tau) \simeq 2\psi_0 \oplus \bigoplus_{t \in \mathbb{F}_q^\times} \psi_t$.
(2) $\chi_{i_B^G \tau} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{cases} q+1, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases}$.

Lemma. *If $g \in G$ has no eigenvalues in $\mathbb{F}_q^\times$, then $\chi_{i_B^G \tau}(g) = 0$.*

Proof. If $\chi_{i_B^G \tau}(g) \neq 0$, then by the Frobenius Character Formula, $\mathrm{cl}_G(g) \cap B \neq \emptyset$. Any element of B has eigenvalues in $\mathbb{F}_q^\times$. qed

Lemma. *Let τ_0 and τ_0' be characters of $\mathbb{F}_q^\times$, with extensions τ and τ' to B. Then $\chi_{i_B^G \tau} = \chi_{i_B^G \tau'}$ if and only if $\tau_0 = (\tau_0')^{\pm 1}$.*

Proof. Note that $\tau_0(-1) = \tau_0^{-1}(-1)$. Hence $\tau_0(-1) = \tau_0'(-1)$ if and only if $(\tau_0 + \tau_0^{-1})(-1) = (\tau_0' + \tau_0'^{-1})(-1)$. Given the above results about $\chi_{i_B^G \tau}$ and $\chi_{i_B^G \tau'}$, we see that $\chi_{i_B^G \tau} = \chi_{i_B^G \tau'}$ if and only if

$$(\tau_0 + \tau_0^{-1})(a) = (\tau_0' + \tau_0'^{-1})(a), \qquad \forall\ a \in \mathbb{F}_q^\times.$$

By linear independence of characters of $\mathbb{F}_q^\times$, this is equivalent to $\tau_0 = \tau_0'^{\pm 1}$. qed

Corollary. *Let* τ *and* τ_0' *be as above Assume that* $\tau_0^2 \neq 1$*. Then*

$$\mathrm{Hom}_G(i_B^G\tau', i_B^G\tau) = \begin{cases} \mathbb{C}, & \text{if } \tau_0' = \tau_0^{\pm 1} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $(\tau_0')^2 = 1$. Then $i_B^G\tau$ is irreducible of degree $q+1$ and $i_B^G\tau'$ is reducible of degree $q+1$. Therefore the two representations do not have any irreducible constituents in common.

Now suppose that $(\tau_0')^2 \neq 1$ Then $i_B^G\tau$ and $i_B^G\tau'$ are irreducible and by the previous result are equivalent if and only if $\tau_0 = \tau_0'^{\pm 1}$. qed

From above, we can conclude that there are have $(q-3)/2$ equivalence classes of irreducible representations of the form $i_B^G\tau$ (of course with $\tau^2 \neq 1$).

The next step is to decompose $i_B^G\tau$ in the two cases with $\tau^2 = 1$.

Lemma. *The trivial representation of* G *occurs as a subrepresentation of the representation* $i_B^G 1$ *induced from the trivial representation of* B*. The other irreducible constituent of* $i_B^G 1$ *has degree* q *and its character is equal to* $\chi_{i_B^G 1}$ *minus the characteristic function of* G*.*

Proof. We already know that $i_B^G 1$ has two irreducible constituents and has degree $q+1$. Therefore it suffices to prove that the trivial representation of G occurs as a subrepresentation of $i_B^G 1$. The characteristic function of G belongs to the space of $i_B^G 1$ and it is clearly invariant under right translation by elements of G. (Although it is not needed for the proof of the lemma, it is worth noting that the space of the other irreducible constituent of $i_B^G 1$ consists of functions which are left B-invariant and which satisfy $f(1) + \sum_{u \in N} f(wu) = 0$). qed

Note that, since we have assumed that p is odd, $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ has order 2. Fix a nonsquare $\varepsilon \in \mathbb{F}_q^\times$. Then 1 and ε are coset representatives for $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. Let λ be the unique character of $\mathbb{F}_q^\times$ of order 2. Let $f_t = f_t^\tau$ (for τ corresponding to λ), $t \in \mathbb{F}_q$, and let $f_B = f_B^\tau$. Let $\rho = i_B^G\tau$. We know that G is generated by N, A and w. Hence a subspace of the space $\mathcal{V}_\lambda$ is G-invariant if and only if it is N, A, and w-invariant. We have already described the N-invariant subspaces. Next, we describe some B-invariant subspaces. Let

$$\mathcal{W}_1 = \mathrm{Span}\{\, f_t \mid t \in (\mathbb{F}_q^\times)^2 \,\} \quad \text{and} \quad \mathcal{W}_\varepsilon = \mathrm{Span}\{\, f_t \mid t \in \varepsilon(\mathbb{F}_q^\times)^2 \,\}.$$

If $x \in \mathbb{F}_q$, let $u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

Lemma. *Let* $a \in \mathbb{F}_q^\times$*.*

(1) If $t \in \mathbb{F}_q^\times$*, then* $\rho(s_a)f_t = \lambda(a)f_{a^{-2}t}$*.*

(2) $\rho(s_a)f_0 = \lambda(a)f_0$

(3) $\rho(s_a)f_B = \lambda(a)f_B$*.*

(4) $\mathcal{W}_1$, $\mathcal{W}_\varepsilon$, $\mathrm{Span}\{f_0\}$, *and* $\mathrm{Span}\{f_B\}$ *are* B*-invariant.*

Proof. Part (4) follows from parts (1)–(3). For parts (1) and (2), let $a \in \mathbb{F}_q^\times$, $t \in \mathbb{F}_q$ and $b \in B$. Then

$$\begin{aligned}(\rho(s_a)f_t)(bwu_x) &= f_t(bwu_x s_a) = f_t(bs_{a^{-1}}wu_{a^{-2}x}) = \lambda(a^{-1})\tau(b)f_t(wu_{a^{-2}x}) \\ &= \lambda(a)\tau(b)\psi_t(a^{-2}x) = \lambda(a)\tau(b)\psi_{a^{-2}t}(x) = \lambda(a)f_{a^{-2}t}(bwu_x).\end{aligned}$$

Above we have used the fact that $bs_a^{-1} \in s_{a^{-1}}bN$ and τ is trivial on N, and $\lambda = \lambda^{-1}$. Part (3) is left as an exercise. qed

Exercise: Prove that there are 4 inequivalent irreducible representations of B that are not one-dimensional, and they all have degree $(q-1)/2$.

Now we consider the action of w. It is easy to see that, as representations of B, $\mathcal{W}_1$ and $\mathcal{W}_\varepsilon$ are irreducible. It follows from results above that they are inequivalent, since as representations of N they have no constituents in common. As we see below, no subspace of $\mathrm{Span}\{\, f_B, f_0 \,\}$ is w-invariant. Hence $\mathcal{W}_1$ and $\mathcal{W}_\varepsilon$ cannot belong to the same G-subrepresentation of $i_B^G\tau$.

We will use

$$wu_x w = s_{-x^{-1}}wu_{-x^{-1}}, \qquad x \in \mathbb{F}_q^\times \tag{3.2}$$

Let $\rho = i_B^G\tau$, where τ corresponds to λ.

Lemma. *Let* $t \in \mathbb{F}_q$.

(1) Then $(\rho(w)f_t)(bwu_x) = f_t(bwu_x w) = \lambda(-x)f_t(bwu_{-x^{-1}}) = \lambda(-x)\psi_t(-x^{-1})$, $x \in \mathbb{F}_q^\times$, $b \in B$,

(2) $(\rho(w)f_t)(bw) = 0$, $b \in B$.

(3) $(\rho(w)f_t)(b) = \tau(b)$, $b \in B$.

Let

$$\langle \varphi_1, \varphi_2 \rangle = \varphi_1(1)\overline{\varphi_2(1)} + \sum_{x\in\mathbb{F}_q} \varphi_1(wu_x)\overline{\varphi_2(wu_x)}, \qquad \varphi_1,\, \varphi_2 \in \mathcal{V}_\tau.$$

Lemma.

(1) $\langle\cdot,\cdot\rangle$ *is an inner product on* $\mathcal{V}_\tau$ *with respect to which* ρ *is a unitary representation of* G.

(2) $\{\, f_B,\, f_t \mid t \in \mathbb{F}_q \,\}$ *is an orthogonal basis of* $\mathcal{V}_\tau$ *(relative to the given inner product).*

Proof. Part (1) is easily verified using the decomposition of G given in the first lemma. For part (2), note that if $t \in \mathbb{F}_q$, the support of f_t does not intersect B. Hence $\langle f_t, f_B\rangle = 0$. Let $s,\, t \in \mathbb{F}_q$. Then

$$\langle f_t, f_s \rangle = \sum_{x\in\mathbb{F}_q} \psi_t(x)\overline{\psi_s(x)} = q\,\delta_{st},$$

using orthogonality relations of characters of $\mathbb{F}_q$. qed

Let $\Gamma = \sum_{x\in\mathbb{F}_q^\times} \lambda(x)\psi(x)$.

Lemma.
(1) $\rho(w)f_B = q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q} f_t$.
(2) $\rho(w)f_0 = f_B + q^{-1}\Gamma\sum_{t\in\mathbb{F}_q^\times} \lambda(t)f_t$.

Proof. For (1), note that

$$q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q} f_t(w) = q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q}\psi_t(0) = \lambda(-1) = f_B(w^2) = (\rho(w)f_B)(w)$$

and, for $x\in\mathbb{F}_q^\times$,

$$q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q} f_t(wu_x) = q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q}\psi_t(x) = q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q}\psi_x(t) = 0 = (\rho(w)f_B)(wu_x),$$

since ψ_x is a nontrivial character of $\mathbb{F}_q$, and $wu_xw^{-1}\notin B$ if $x\in\mathbb{F}_q^\times$.

From the previous lemma, we have $\rho(w)f_0 = f_B + \sum_{t\in\mathbb{F}_q} c_tf_t$ for some scalars c_t. And because $\{\, f_B,\ f_t \mid t\in\mathbb{F}_q\,\}$ is an orthogonal basis of $\mathcal{V}_\tau$, we have $c_t = \langle\rho(w)f_0, f_t\rangle/\langle f_t, f_t\rangle = q^{-1}\langle\rho(w)f_0, f_t\rangle$, $t\in\mathbb{F}_q$. Because $f_t(1) = (\rho(w)f_0)(w) = f_0(-1) = 0$, we have

$$\begin{aligned}\langle\rho(w)f_0, f_t\rangle &= \sum_{x\in\mathbb{F}_q^\times}\rho(w)f_0(wu_x)\overline{f_t(wu_x)} = \sum_{x\in\mathbb{F}_q^\times}\lambda(-x)\psi_0(-x^{-1})\overline{\psi_t(x)} = \sum_{x\in\mathbb{F}_q^\times}\lambda(-x)\psi_t(-x)\\ &= \begin{cases}\sum_{x\in\mathbb{F}_q^\times}\lambda(t^{-1}x)\psi(x) = \lambda(t)\sum_{x\in\mathbb{F}_q^\times}\lambda(x)\psi(x) = \lambda(t)\Gamma, & \text{if } t\in\mathbb{F}_q^\times\\ \sum_{x\in\mathbb{F}_q^\times}\lambda(x) = 0, & \text{if } t = 0\end{cases}\end{aligned}$$

Lemma. $\Gamma^2 = q\lambda(-1)$.

Proof. Let $\Phi(t) = \sum_{x\in\mathbb{F}_q^\times}\psi(tx)\lambda(x)$, $t\in\mathbb{F}_q^\times$. Then

$$\Phi(t) = \sum_{y\in\mathbb{F}_q^\times}\psi(y)\lambda(t)^{-1}\lambda(y) = \lambda(t)\Gamma.$$

Then, if $x\in\mathbb{F}_q^\times$,

$$\sum_{t\in\mathbb{F}_q^\times}\Phi(t)\psi(tx) = \Gamma\sum_{t\in\mathbb{F}_q^\times}\lambda(t)\psi(tx) = \Gamma\Phi(x) = \Gamma^2\lambda(x).$$

But we also have

$$\begin{aligned}\sum_{t\in\mathbb{F}_q^\times}\Phi(t)\psi(tx) &= \sum_{t\in\mathbb{F}_q^\times}\sum_{u\in\mathbb{F}_q^\times}\psi(tu+tx)\lambda(u) = \sum_{u\in\mathbb{F}_q^\times}\left(\sum_{t\in\mathbb{F}_q^\times}\psi(t(u+x))\right)\lambda(u)\\ &= \lambda(-x)(q-1) - \sum_{u\in\mathbb{F}_q^\times,\, u\neq -x}\lambda(u) = \lambda(-x)(q-1) - (0-\lambda(-x)) = \lambda(-x)q\end{aligned}.$$

Therefore $\Gamma^2\lambda(x) = \lambda(-x)q$. This implies $\Gamma^2 = \lambda(-1)q$. qed

Lemma.

(1) $\rho(w)(\Gamma f_0 \pm q f_B) = \pm q^{-1}\Gamma(\Gamma f_0 \pm q f_B) + \lambda(-1)\sum_{t\in\mathbb{F}_q^\times}(\lambda(t)\pm 1)f_t$.

(2) $\rho(w)(\Gamma f_0 + q f_B) \in \mathrm{Span}(\Gamma f_0 + q f_B) + \mathcal{W}_1$.

(3) $\rho(w)(\Gamma f_0 - q f_B) \in \mathrm{Span}(\Gamma f_0 - q f_B) + \mathcal{W}_\varepsilon$.

Set $\mathcal{V}^+ = \mathrm{Span}(\Gamma f_0 + q f_B) + \mathcal{W}_1$ and $\mathcal{V}^- = \mathrm{Span}(\Gamma f_0 - q f_B) + \mathcal{W}_\varepsilon$. We can see that $\mathcal{V}^+ = (\mathcal{V}^-)^\perp$ and $\mathcal{V}^- = (\mathcal{V}^+)^\perp$. The above lemma suggests that $\mathcal{V}^+$ and $\mathcal{V}^-$ may be G-invariant. We need to show that $\rho(w)\mathcal{W}_1 \subset \mathcal{V}^+$ and $\rho(w)\mathcal{W}_\varepsilon \subset \mathcal{V}^-$.

Lemma.

(1) Let $s,\, t \in \mathbb{F}_q^\times$. *Then* $\langle \rho(w)f_t, f_s\rangle = 0$ *if* $s \notin t(\mathbb{F}_q^\times)^2$.

(2) If $t \in (\mathbb{F}_q^\times)^2$, *then* $\langle \rho(w)f_t, \Gamma f_0 - q f_B\rangle = 0$.

(3) If $t \in \varepsilon(\mathbb{F}_q^\times)^2$, *then* $\langle \rho(w)f_t, \Gamma f_0 + q f_B\rangle = 0$.

Proof. Let $s,\, t \in \mathbb{F}_q^\times$. Then

$$\langle \rho(w)f_t, f_s\rangle = \sum_{x\in\mathbb{F}_q^\times} \rho(w)f_t(wu_x)\overline{f_s(wu_x)} = \sum_{x\in\mathbb{F}_q^\times} \lambda(x)\psi_t(x^{-1})\psi_s(x) = \sum_{x\in\mathbb{F}_q^\times} \lambda(x)\psi(tx^{-1}+sx).$$

Now suppose that $s = \varepsilon t u^2$ for some $u \in \mathbb{F}_q^\times$. Then

$$\sum_{x\in\mathbb{F}_q^\times} \lambda(x)\psi(t^{-1}x + sx) = \sum_{x\in(\mathbb{F}_q^\times)^2} \psi_t(x^{-1} + x\varepsilon u^2) - \sum_{x\in(\mathbb{F}_q^\times)^2} \psi_t(\varepsilon x^{-1} + xu^2).$$

The map $x \mapsto xu^2$ is a bijection from $\{\, x^{-1} + x\varepsilon u^2 \mid x \in (\mathbb{F}_q^\times)^2 \,\}$ and $\{\, \varepsilon x^{-1} + xu^2 \mid x \in (\mathbb{F}_q^\times)^2 \,\}$. It follows that $\langle \rho(w)f_t, f_{\varepsilon t u^2}\rangle = 0$ for all $u \in \mathbb{F}_q^\times$.

For (2), let $t \in (\mathbb{F}_q^\times)^2$. Then

$$\begin{aligned}\langle \rho(w)f_t, \Gamma f_0 - q f_B\rangle &= \langle f_t, \rho(-w)(\Gamma f_0 - q f_B)\rangle \\ &= \lambda(-1)\langle f_t, -\Gamma q^{-1}(\Gamma f_0 - q f_B) + \lambda(-1)\sum_{s\in\varepsilon(\mathbb{F}_q^\times)^2} -2f_s\rangle = 0.\end{aligned}$$

The proof of part (3) is similar to that of part (2) and is left as an exercise. qed

Corollary. *$\mathcal{V}^+$ and $\mathcal{V}^-$ are G-invariant.*

Let $\pi_\lambda^+ = \rho\,|_{\mathcal{V}^+}$ and $\pi_\lambda^- = \rho\,|_{\mathcal{V}^-}$.

For the proof of the following lemma, see the discussion on conjugacy classes which appears on pages 31.

Lemma. *Let* $x \in \mathbb{F}_q^\times$.

(1) $|\mathrm{cl}_G(u_x)| = (q^2-1)/2$.

(2) $\mathrm{cl}_G(u_x) = \mathrm{cl}_G(u_1)$ *if* $x \in (\mathbb{F}_q^\times)^2$.

(3) $\mathrm{cl}_G(u_x) = \mathrm{cl}_G(u_\varepsilon)$ *if* $x \in \varepsilon(\mathbb{F}_q^\times)^2$.
(4) $\mathrm{cl}_G(u_1) \neq \mathrm{cl}_G(u_\varepsilon)$.

Theorem. $\rho = \pi_\lambda^+ \oplus \pi_\lambda^-$, π_λ^+ *and* π_λ^- *are irreducible and inequivalent. Furthermore,*
(1) $\chi_{\pi_\lambda^+}(s_{\pm 1}) = \lambda(-1)(q+1)/2$.
(2) $\chi_{\pi_\lambda^+}(s_a) = \lambda(a)$, $a \in \mathbb{F}_q^\times$, $a^2 \neq 1$.
(3) $\chi_{\pi_\lambda^+}(\pm u_1) = \lambda(-1)(1+\Gamma)/2$.
(4) $\chi_{\pi_\lambda^+}(\pm u_\varepsilon) = \lambda(-1)(1-\Gamma)/2$.
(5) $\chi_{\pi_\lambda^+}(g) = 0$ *if* g *has no eigenvalues in* $\mathbb{F}_q^\times$.
(6) $\chi_{\pi_\lambda^-} = \chi_\rho - \chi_{\pi_\lambda^+} = \chi_{i_B^G \tau} - \chi_{\pi_\lambda^+}$ *(where* τ *is the character of* B *corresponding to* λ*).*

Proof. Because the restrictions of π_λ^+ and π_λ^- to N are inequivalent, the representations are inequivalent. For part (1), note that $\dim \mathcal{V}^+ = (q+1)/2$ and $\rho(-1) = \lambda(-1)I$.

For part (2), recall that $\rho(s_a)f_t = \lambda(a) f_{a^{-2}t}$, $t \in (\mathbb{F}_q^\times)^2$. Therefore $\mathrm{tr}\,(\rho(s_a)\,|_{\mathcal{W}_1}) = 0$ if $a^2 \neq 1$. Also, $\rho(s_a)(\Gamma f_0 + q f_B) = \lambda(a)(\Gamma f_0 + q f_B)$, $a \in \mathbb{F}_q^\times$.

Let $x \in \mathbb{F}_q$. Set

$$h_1(x) = \sum_{t \in (\mathbb{F}_q^\times)^2} \psi_t(x) = \sum_{t \in (\mathbb{F}_q^\times)^2} \psi_x(t) \quad \text{and} \quad h_\varepsilon(x) = \sum_{t \in \varepsilon(\mathbb{F}_q^\times)^2} \psi_t(x) = \sum_{t \in \varepsilon(\mathbb{F}_q^\times)^2} \psi_x(t).$$

If $x \in \mathbb{F}_q^\times$, then $h_1(x) + h_\varepsilon(x) = -1$ because ψ_x is a nontrivial character of $\mathbb{F}_q$. Also, if $x \in \mathbb{F}_q^\times$,

$$h_1(x) - h_\varepsilon(x) = \sum_{t \in \mathbb{F}_q^\times} \lambda(t)\psi_x(t) = \sum_{s \in \mathbb{F}_q^\times} \lambda(x^{-1}s)\psi(s) = \lambda(x)\Gamma.$$

Solving for $h_1(x)$ and $h_\varepsilon(x)$, we get $h_1(x) = (-1+\lambda(x)\Gamma)/2$ and $h_\varepsilon(x) = (-1-\lambda(x)\Gamma)/2$, $x \in \mathbb{F}_q^\times$. Now $\mathrm{tr}\,(\rho(u_x)\,|_{\mathcal{W}_1}) = h_1(x)$ and $\rho(u_x)(\Gamma f_0 + q f_B) = \Gamma f_0 + q f_B$. Therefore $\chi_{\pi_\lambda^+}(u_1) = h_1(1) + 1 = (1+\Gamma)/2$. Similarly, $\chi_{\pi_\lambda^+}(u_\varepsilon) = h_1(\varepsilon) + 1 = (1-\Gamma)/2$. Parts (3) and (4) follow.

Let G_{spl} be the set of elements of G which have eigenvalues in $\mathbb{F}_q^\times$. Any element in G_{spl} is conjugate to one of s_a, $a \in \mathbb{F}_q^\times$ or one of $\pm u_1$ and $\pm u_\varepsilon$. Therefore

$$\begin{aligned}\sum_{g \in G_{spl}} |\chi_{\pi_\lambda^+}(g)|^2 &= 2((q+1)/2)^2 + q(q+1)(q-3)/2 + 2((1+\Gamma+\bar\Gamma+\Gamma\bar\Gamma)/4)(q^2-1)/2 \\ &\quad + 2((1-\Gamma-\bar\Gamma+\Gamma\bar\Gamma)/4)(q^2-1)/2 \\ &= (q+1)^2/2 + q(q+1)(q-3)/2 + (1+\Gamma\bar\Gamma)(q^2-1)/2\end{aligned}$$

Now

$$\bar\Gamma = \sum_{x \in \mathbb{F}_q^\times} \lambda(x)\psi(-x) = \lambda(-1)\sum_{x \in \mathbb{F}_q^\times} \lambda(x)\psi(x) = \lambda(-1)\Gamma.$$

Hence $\Gamma\bar{\Gamma} = \lambda(-1)\Gamma^2 = \lambda(-1)^2 q = q$. Substituting above results in $\sum_{g\in G_{spl}} |\chi_{\pi_\lambda^+}(g)|^2 = q(q^2-1) = |G|$. Because π_λ^+ is irreducible, we have $\sum_{g\in G} |\chi_{\pi_\lambda^+}(g)|^2 = |G|$. Therefore $\chi_{\pi_\lambda^+}(g) = 0$ if $g \notin G_{spl}$ qed

Before moving on to finding the other irreducible characters of G, we discuss conjugacy classes further. Let $g \in G$ be unipotent (that is, $(g-1)^2 = 0$). Then both eigenvalues of g equal 1. Suppose that $g \neq 1$ and g is unipotent. Then, because u_1 is the Jordan canonical form of g, there exists $g_0 \in GL_2(\mathbb{F}_q)$ such that $g_0 g g_0^{-1} = u_1$. That is, there is one noncentral unipotent conjugacy class in $GL_2(\mathbb{F}_q)$. This class breaks up into several conjugacy classes in G. The centralizer of u_1 in G is $\pm N$, so $|\mathrm{cl}_G(u_1)| = (q^2-1)/2$. From $s_a u_x s_a = u_{a^2 x}$, $x \in \mathbb{F}_q^\times$, we see that $u_{a^2} \in \mathrm{cl}_G(u_1)$. Let $d(\varepsilon)$ be the diagonal matrix in $GL_2(\mathbb{F}_q)$ having diagonal entries ε and 1, respectively. If $u_\varepsilon \in \mathrm{cl}_G(u_1)$, then, choosing $g \in G$ such that $gu_1g^{-1} = i_\varepsilon$, we have $d(\varepsilon)u_1 d(\varepsilon)^{-1} = gu_1g^{-1}$. This implies $g^{-1}d(\varepsilon)$ centralizes u_1. Therefore $g^{-1}d(\varepsilon) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ for some $a \in F_q^\times$ and some $b \in \mathbb{F}_q$. Taking determinants, we have $\varepsilon = a^2$, which is impossible, since ε is a non-square. Hence $\mathrm{cl}_G(u_1) \neq \mathrm{cl}_G(u_\varepsilon)$.

Now suppose that $g \in G_{ell}$. Then the eigenvalues of g, being roots of the characteristic polynomial of g are roots of a quadratic polynomial which is irreducible over $\mathbb{F}_q$. Hence the eigenvalues lie in a quadratic extension of $\mathbb{F}_q$. Up to isomorphism, there is exactly one quadratic extension of $\mathbb{F}_q$. Since ε is a non-square in $\mathbb{F}_q^\times$, $\mathbb{F}_q(\sqrt{\varepsilon})$ is a quadratic extension of $\mathbb{F}_q$. Hence the eigenvalues of g lie in $\mathbb{F}_q(\sqrt{\varepsilon}) - \mathbb{F}_q$. Now $g \in G$, so the product of the eigenvalues equals 1. It follows that the eigenvalues of g are of the form $a + b\sqrt{\varepsilon}$ and $a - b\sqrt{\varepsilon}$ where $a \in \mathbb{F}_q$, $b \in \mathbb{F}_q^\times$, and $a^2 - b^2\varepsilon = 1$. An example of such a g is the matrix $\begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix}$, with $b \neq 0$ and $a^2 - b^2\varepsilon = 1$. Let

$$T = \left\{ \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \mid a,\, b \in \mathbb{F}_q,\ a^2 - b^2\varepsilon = 1 \right\}.$$

Then T is a subgroup of G and any element of $T - \{\pm I\}$ belongs to G_{ell}. Note that $\{1, \varepsilon\}$ is a basis of $\mathbb{F}_q(\sqrt{\varepsilon})$, and the matrix of an elment $a + b\sqrt{\varepsilon}$ with respect to this basis is equal to $\begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix}$. The map $\mathcal{N} : \mathbb{F}_q(\sqrt{\varepsilon})^\times \longrightarrow \mathbb{F}_q^\times$ is a surjective homomorphism. Hence the kernel of $\mathcal{N}$ has order $(q^2-1)/(q-1) = q+1$. The map $a + b\sqrt{\varepsilon} \mapsto \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix}$ from $\mathbb{F}_q(\sqrt{\varepsilon})^\times$ to $GL_2(\mathbb{F}_q)$ restricts to an isomorphism between $\mathcal{N}$ and T. Hence $|T| = q+1$. Also, T is cyclic, since it is a subgroup of the cyclic group $\mathbb{F}_q(\sqrt{\varepsilon})^\times \simeq \mathbb{F}_{q^2}^\times$. It is a simple matter to check that if $\gamma \in T - \{\pm I\}$, then the centralizer of γ in G is equal to T. Hence $|\mathrm{cl}_G(\gamma)| = q(q-1)$. Now let $\gamma = \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \in T$ be such that $\gamma^2 \neq 1$. Any element of T which is conjugate to γ must have the same eigenvalues as γ, namely $a + b\sqrt{\varepsilon}$ and $a - b\sqrt{\varepsilon} = (a + b\sqrt{\varepsilon})^{-1}$. Checking that γ and γ^{-1} are conjugate in G, we have $\mathrm{cl}_G(\gamma) \cap T = \{\gamma, \gamma^{-1}\}$. There are $q - 1$ noncentral elements in T. Thus there are exactly $(q - 1)/2$ noncentral conjugacy classes which intersect T, each containing $q(q - 1)$ elements.

Proposition.

1. *The centre of G equals $\{\pm I\}$.*
2. *If $a \in \mathbb{F}_q^\times$ and $a \neq \pm 1$, then $|cl_G(s_a)| = q(q+1)$. There are exactly $(q-3)/2$ noncentral conjugacy classes which intersect A.*
3. *$\mathrm{cl}(u_1) \neq \mathrm{cl}(u_\varepsilon)$, $\mathrm{cl}(-u_1) \neq \mathrm{cl}(-u_\varepsilon)$, and $|\mathrm{cl}(\pm u_\varepsilon)| = |\mathrm{cl}(\pm u_1)| = (q^2 - 1)/2$.*
4. *If $\gamma \in T$ and $\gamma \neq \pm 1$, then $|cl(\gamma)| = q(q - 1)$. There are exactly $(q - 1)/2$ noncentral conjugacy classes which intersect T.*
5. *There are $q + 4$ conjugacy classes in G.*

Proof. The only part which has not already been proved is part (5). First, counting the conjugacy classes mentioned in parts (1)–(4), we have a total of

$$2 + (q - 3)/2 + 4 + (q - 1)/2 = q + 4.$$

Counting the number of elements in the unions of all of these conjugacy classes, we get

$$2 + q(q + 1)(q - 3)/2 + 4(q^2 - 1)/2 + q(q - 1)(q - 1)/2 = q(q^2 - 1) = |G|.$$

The number of distinct irreducible characters of G constructed so far (from the representations $i_B^G\tau$ and their irreducible constituents) is equal to $(q - 3)/2 + 4 = (q + 5)/2$, $(q-3)/2$ of degree $q+1$, one of degree 1, one of degree q, and two of degree $(q+1)/2$. We must find another $(q + 3)/2$ irreducible characters. We have found the characters of those irreducible representations of G whose restrictions to N contain the trivial representations of N (equivalently whose restrictions to B contain a one-dimensional representation of B). An irreducible representation of G is said to be *cuspidal* if its restriction to N does not contain the trivial representation of N (equivalently, its restriction to B contains no one-dimensional representations of B).

As we will see, the nontrivial characters of the group T will determine the other irreducible characters of G, but not in exactly the same way that the characters of A determined the characters already constructed.

Exercise: For each $x \in \mathbb{F}_q$, define

$$\psi^+(\pm u_x) = \psi(x), \quad \psi^-(\pm u_x) = \pm\psi(x)$$
$$\psi_\varepsilon^+(\pm u_x) = \psi_\varepsilon(x), \quad \psi_\varepsilon^-(\pm u_x) = \pm\psi_\varepsilon(x).$$

Let $\sigma_1^\pm = i_{\pm N}^B \psi^\pm$, and $\sigma_\varepsilon^\pm = i_{\pm N}^B \psi_\varepsilon^\pm$. Prove that

(1) $\sigma_1^\pm$ and $\sigma_\varepsilon^\pm$ are inequivalent irreducible representations of B, and any irreducible representation of B of degree greater than 1 is equivalent to one of them.
(2) $\chi_{\sigma_1^\pm}(\pm u_1) = \chi_{\sigma_\varepsilon^\pm}(\pm u_\varepsilon) = \pm(\Gamma - 1)/2$
(3) $\chi_{\sigma_1^\pm}(\pm u_\varepsilon) = \chi_{\sigma_\varepsilon^\pm}(\pm u_1) = \pm(-\Gamma - 1)/2$.
(4) All four of the above characters vanish on $B - \pm N$.

A class function φ on a finite group G' is a *virtual character* of G' if φ is an integral linear combination of characters of irreducible representations of G. Because the sum and product of the characters of representations of G' are also characters of representations (the product being the character of a tensor product), the set of virtual characters of G' is a ring (with pointwise addition and multiplication of functions). Let $\chi_1, \dots, \chi_r$ be the characters of a complete set of irreducible representations of G'. Suppose that $\varphi = \sum_{j=1}^r \ell_j \chi_j$. Then $\langle \varphi, \varphi \rangle_{\mathcal{A}(G')} = \sum_{j=1}^r \ell_j^2$. It follows that if $\langle \varphi, \varphi \rangle = 1$, then $\varphi = \pm \chi_j$ for some j. If, in addition, $\varphi(1) > 0$, then $\varphi = \chi_j$. Note that we cannot take this approach with arbitrary class functions on G because we can find complex numbers $c_1, \dots, c_r$ with $\sum_{j=1}^r |c_j|^2 = 1$ without forcing $\sum_{j=1}^r c_j \chi_j$ to be a multiple of one χ_j. One approach to determining irreducible characters of a finite group G' is to generate as many virtual characters φ as possible, and then look for those which satisfy $\langle \varphi, \varphi \rangle_{\mathcal{A}(G)} = 1$.

Let π be an irreducible cuspidal representation of G. Because the restriction of π to the subgroup B is the direct sum of irreducible representations of B of degree greater than 1, and every irreducible representation of B has degree $(q-1)/2$, $\chi_\pi(1)$ is divisible by $(q-1)/2$. By Frobenius reciprocity, π occurs as a subrepresentatation of $i_B^G \sigma_1^\pm = i_{\pm N}^G \psi^\pm$ or $i_B^G \sigma_\varepsilon^\pm = i_{\pm N}^G \psi_\varepsilon^\pm$. The proof of the following lemma is left as an exercise.

Lemma.
(1) $\chi_{i_{\pm N}^G \psi^\pm}(\pm 1) = \chi_{i_{\pm N}^G \psi_\varepsilon^\pm}(\pm 1) = \pm(q^2 - 1)/2.$
(2) $\chi_{i_{\pm N}^G \psi^\pm}(\pm u_1) = \pm(\Gamma - 1)/2$
(3) $\chi_{i_{\pm N}^G \psi^\pm}(\pm u_\varepsilon) = \pm(-\Gamma - 1)/2$
(4) $\chi_{i_{\pm N}^G \psi_\varepsilon^\pm}(\pm u_1) = \pm(-\Gamma - 1)/2$
(5) $\chi_{i_{\pm N}^G \psi_\varepsilon^\pm}(\pm u_\varepsilon) = \pm(\Gamma - 1)/2$
(6) $\chi_{i_{\pm N}^G \psi_\varepsilon^\pm}$ *and* $\chi_{i_{\pm N}^G \psi^\pm}$ *vanish on elements* $s_a \in A$ *with* $a^2 \neq 1$ *and* $\gamma \in T$ *with* $\gamma^2 \neq 1$.

The irreducible characters constructed so far as all constant on G_{ell}. Therefore the restrictions of the characters of the cuspidal representations of G must separate the conjugacy classes $\mathrm{cl}(\gamma)$, $\gamma \in T$ such that $\gamma^2 \neq 1$. From the above lemma, we see that the characters $\chi_{i_{\pm N}^G \psi^\pm}$ and $\chi_{i_{\pm N}^G \psi_\varepsilon^\pm}$ vanish on G_{ell}, so the characters of the elliptic reprsentations cannot all be expressed as linear combinations of these characters. The nontrivial characters of T do separate these conjugacy classes in G_{ell}. So perhaps the characters of the representations $i_T^G \theta$ for θ ranging over nontrivial characters of T will be related to the characters of the cuspidal representations of G. The proof of the following lemma is left as an exercise.

Lemma. *Let θ be a character of T. Then*

(1) $\chi_{i_T^G\theta}(\pm 1) = \theta(\pm 1)q(q-1)$

(2) $\chi_{i_T^G\theta}(\gamma) = \theta(\gamma) + \theta(\gamma^{-1})$ *if* $\gamma \in T$ *and* $\gamma^2 \neq 1$.

(3) Let $a \in \mathbb{F}_q^\times$ *be such that* $a^2 \neq 1$. *Then* $\chi_{i_T^G\theta}(s_a) = \chi_{i_T^G}(\pm u_1) = \chi_{i_T^G}(\pm u_\varepsilon) = 0$.

Since the functions $\chi_{i_{\pm N}^G\psi^\pm}$, $\chi_{i_{\pm N}^G\psi_\varepsilon^\pm}$ and $\chi_{i_T^G\theta}$ are virtual characters of G, so is any integral linear combination of these functions. We will look for virtual characters φ of this form which satisfy $\langle \varphi, \varphi \rangle = 1$.

Suppose that π is an irreducible representation of G. Since $-I$ belongs to the centre of G, it follows from Schur's Lemma that $\pi(-I)$ is a scalar multiple of the identity operator. And $(-I)^2 = I$ forces the scalar multiple to equal ± 1. Looking back at the irreducible representations of the form $i_B^G\tau$, on the unipotent set, the character formula involves the trivial character of N (appearing twice) and sum $\chi_{i_{\pm N}^G\psi^+} + \chi_{i_{\pm N}^G\psi_\varepsilon^+}$ or $\chi_{i_{\pm N}^G\psi^-} + \chi_{i_{\pm N}^G\psi_\varepsilon^-}$, with $+$ if the above scalar is -1 and $-$ if the scalar is -1 (note that this scalar equals $\tau(-1)$). On noncentral elements of of A, the sum $\tau + \tau^{-1}$ appears. With this in mind, knowing that the trivial representation of N will not occur in the character of a cuspidal representation, suppose that θ is a character of T, and set

$$\varphi_\theta = \begin{cases} \ell\chi_{i_T^G\theta} + m(\chi_{i_{\pm N}^G\psi^+} + \chi_{i_{\pm N}^G\psi_\varepsilon^+}, & \text{if } \theta(-1) = 1 \\ \ell\chi_{i_T^G\theta} + m(\chi_{i_{\pm N}^G\psi^-} + \chi_{i_{\pm N}^G\psi_\varepsilon^-}, & \text{if } \theta(-1) = -1. \end{cases}$$

where $m, \ell \in \{\pm 1\}$. Now if φ_θ is the character of a cuspidal representation, we must have $\varphi_\theta(1) = \ell q(q-1) + m(q^2-1) = (q-1)(\ell q + m(q+1)) > 0$, equal to a multiple of $(q-1)/2$, and dividing $q(q^2-1) = |G|$. Checking the possibilities, we must have $m = 1$ and $\ell = -1$, so $\varphi_\theta(\pm 1) = \theta(-1)(q-1)$.

The values of φ_θ on $\pm u_\varepsilon$, $\pm u_1$, $\gamma \in T$ and s_a, $a \in \mathbb{F}_q^\times$ may be obtained from character values given in the previous two lemmas.

Lemma.

(1) $\varphi_\theta(\pm 1) = \theta(-1)(q-1)$

(2) $\varphi_\theta(\pm u_1) = \varphi_\theta(\pm u_\varepsilon) = -\theta(\pm 1)$.

(3) If $\gamma \in T$ *and* $\gamma^2 \neq 1$, *then* $\varphi_\theta(\gamma) = -\theta(\gamma) - \theta(\gamma^{-1})$.

(4) If $a \in \mathbb{F}_q^\times$ *and* $a^2 \neq 1$, *then* $\varphi_\theta(s_a) = 0$.

Lemma.

$$\langle \varphi_\theta, \varphi_\theta \rangle_{\mathcal{A}(G)} = \begin{cases} 1, & \textit{if } \theta^2 \neq 1 \\ 2, & \textit{if } \theta^2 = 1 \end{cases}.$$

Proof. According to the above lemma, $|\varphi_\theta(\pm u_1)|^2 = |\varphi_\theta(\pm u_\varepsilon)|^2 = 1$. Thus

$$
\begin{aligned}
q(q^2-1)\langle \varphi_\theta, \varphi_\theta\rangle_{\mathcal{A}(G)} &= \sum_{g\in G} \varphi_\theta(g)\overline{\varphi_\theta(g)} \\
&= 2(q-1)^2 + 4(q^2-1)/2 + q(q-1)(1/2) \sum_{\gamma\in T,\, \gamma\neq\pm1} (\theta(\gamma)+\theta(\gamma^{-1}))^2 \\
&= 4q(q-1) + q(q-1)/2 \sum_{\gamma\in T\ \gamma^2\neq\pm1} (\theta(\gamma)^2 + 2 + \theta(\gamma^{-1})^2) \\
&= 4q(q-1) + q(q-1)\left((\sum_{\gamma\in T} \theta(\gamma)^2) - 2 + (q-1) \right) \\
&= 4q(q-1) + q(q-1) \begin{cases} q-3, & \text{if } \theta^2 \neq 1 \\ 2(q-1), & \text{if } \theta^2 = 1. \end{cases} \\
&= \begin{cases} q(q^2-1), & \text{if } \theta^2 \neq 1 \\ 2q(q^2-1), & \text{if } \theta^2 = 1. \end{cases}
\end{aligned}
$$

Theorem. *Let θ be a character of T.*

(1) If $\theta^2 \neq 1$, there exists an irreducible representation π_θ of G such that $\chi_{\pi_\theta} = \varphi_\theta$. Also, If θ' is a character of T with nontrivial square, then $\pi_\theta \simeq \pi_{\theta'}$ if and only if $\theta' = \theta^{\pm1}$. Also π_θ is not equivalent to any of the irreducible representations obtained as irreducible constituents of representations induced from one-dimensional representations of B.

(2) Let θ be a character of T such that $\theta^2 = 1$. Then there exist irreducible representations π_θ^+ and π_θ^- of G such that $\varphi_\theta = \chi_{\pi_\theta^+} \pm \chi_{\pi_\theta^-}$.

Proof. Apply the previous result, together with comments on properties of virtual characters, to obtain the proofs of the assertions regarding irreducibility of π_θ when $\theta^2 \neq 1$.

From the values of $\chi_{\pi_\theta} = \varphi_\theta$, we can see that $\chi_{\pi_\theta} = \chi_{\pi_{\theta'}}$ if and only if the functions $\theta + \theta^{-1}$ and $\theta' + (\theta')^{-1}$ agree on T. Hence, by linear independence of characters of T, we have $\pi_\theta \simeq \pi_{\theta'}$ if and only if $\theta' = \theta^{\pm1}$. Clearly the character π_θ is distinct from the characters of any of the irreducible constituents of the representations $i_B^G\tau$.

For (2), assume that $\theta^2 = 1$. We know that φ_θ is a virtual character. It follows from $\langle\varphi_\theta, \varphi_\theta\rangle = 2$ that 2 equals the sums of squares of the integers occuring as coefficients in the expression of φ_θ as a linear combination of irreducible characters. Hence exactly 2 of the coefficients are nonzero, each one in the set $\{\pm1\}$. Now $\varphi_\theta(1) > 0$. So it is clear that π_θ^+ and π_θ^- can be chosen so that the signs must be as stated, qed

At this point, ignoring the representations in part (2) above (whose characters we have not yet computed), we have produced $(q-1)/2$ irreducible characters of the form $i_B^G\tau$, 2 irreducible characters of constitutents of $i_B^G 1$, $\chi_{\pi_\lambda^+}$ and $\chi_{\pi_\lambda^-}$, and $(q-1)/2$ inequivalent irreducible characters χ_{π_θ}. That is, we have produced $q+2$ distinct irreducible characters of G. There are $(q+4)-(q+2) = 2$ remaining to find (and both are cuspidal).

Computing the sums of the squares of the degrees of the irreducible characters already produced, we obtain $|G|-(q-1)^2/2$. Therefore, if d_1 and d_2 are the degrees of the remaining irreducible characters, we have $d_1^2 + d_2^2 = (q-1)^2/2$. We already know that d_1 and d_2 are divisible by $(q-1)/2$ (the degree of any irreducible representation of B that is not one-dimensional). Therefore $d_1 = d_2 = (q-1)/2$.

Suppose that θ is a character of T such that $\theta^2 = 1$. Let π_θ^+ and π_θ^- be as above. Suppose that one of them is not cuspidal (that is, one of them contains a one-dimensional representation of B). Then, because $r_G^B\varphi_\theta$ is a combination of the characters of B of degree $(q-1)/2$, the other one must also be non-cuspidal, and $\varphi_\theta = \chi_{\pi_\theta^+} - \chi_{\pi_\theta^-}$. Furthermore, at least one of π_θ^+ and π_θ^- must have a character which is nonvanishing on G_{ell}, because φ_θ is not nonzero on G_{ell}. As we have seen, there are exactly two equivalence classes of non-cuspidal irreducible representations of G whose characters are not identically zero on G_{ell}. They are the two constitutents of $i_B^G 1$. One is the trivial representation of G, and the other has degree q. Every other non-cuspidal irreducible representation has degree $q+1$. Now $\varphi_\theta(1) = \chi_{\pi_\theta^+}(1) = \chi_{\pi_\theta^-}(1) = q-1$. It follows that π_θ^+ has degree q and π_θ^- is the trivial representation of G. Looking at the values of the characters of these representations, and of φ_θ, we see that we must have θ trivial.

Lemma. *Let θ be the trivial character of T. Then φ_θ is the difference of the irreducible character of degree q and the trivial representation of G.*

Now let ν be the unique character of T of order 2. So ν takes the value -1 on non-squares in T, and 1 on squares in T. According to the comments above, we know that both π_ν^+ and π_ν^- are cuspidal. Now they must have degree a multiple of $(q-1)/2$. From the form of φ_ν, we see that it may be the case that $\varphi_\nu = \chi_{\pi_\nu^+} - \chi_{\pi_\nu^-}$ where $\chi_{\pi_\nu^+}(1) = q-1$ and $\chi_{\pi_\nu^-}(1) = (q-1)/2$. In that case, since we know that the only cuspidal representations which are not equivalent some π_θ with $\theta^2 \neq 1$ all have degree $(q-1)/2$, it follows that $\pi_\nu^+ \simeq \pi_\theta$ for some θ with $\theta^2 \neq 1$. But we can check that $\langle \varphi_\nu, \varphi_\theta \rangle_{\mathcal{A}(G)} = 0$ if $\theta^2 \neq 1$. Therefore it is not possible for $\chi_{\pi_\nu^+}$ to have degree $q-1$.

Proposition. *Let ν be the character of T which has order 2. Then there exist two irreducible inequivalent cuspidal representations π_ν^+ and π_ν^+, both of degree $(q-1)/2$, such that $\varphi_\nu = \chi_{\pi_\nu^+} + \chi_{\pi_\nu^-}$. Furthermore,*
(1) $\chi_{\pi_\nu^+}(\pm u_1) = \chi_{\pi_\nu^-}(\pm u_\varepsilon) = \nu(\pm 1)(-1+\Gamma)/2$.
(2) $\chi_{\pi_\nu^-}(\pm u_1) = \chi_{\pi_\nu^+}(u_\varepsilon) = \nu(\pm 1)(-\Gamma-1)/2$.
(3) If $\gamma \in T$ and $\gamma^2 \neq 1$, then $\chi_{\pi_\nu^+}(\gamma) = \chi_{\pi_\nu^-}(\gamma) = \nu(\gamma)$.
(4) If $a \in \mathbb{F}_q^\times$ and $a^2 \neq 1$, then $\chi_{\pi_\nu^+}(s_a) = \chi_{\pi_\nu^-}(s_a) = 0$.

Proof. We know that $r_G^B(\chi_{\pi_\nu^+})$ and $r_G^B(\chi_{\pi_\nu^-})$ are irreducible characters of B having the property that $r_G^B(\chi_{\pi_\nu^+} + \chi_{\pi_\nu^+})$ equals $\chi_{\sigma_1^+} + \chi_{\sigma_\varepsilon^+}$ if $\nu(-1) = 1$ and $\chi_{\sigma_1^-} + \chi_{\sigma_\varepsilon^-}$ if $\nu(-1) = -1$. This is enought to obtain parts (1) and (2).

Part (3) can be proved using orthogonality relations.

For part (4), since π_ν^+ and π_ν^- are cuspidal, their restrictions to B don't contain any degree one representation of B. As we have already mentioned, the character of an irreducible representation of B of degree $(q-1)/2$ vanishes on elements of the form s_a, $a^2 \neq 1$. qed

Theorem. *Let ψ be a nontrivial character of F. Let λ and ν be the unique characters of $\mathbb{F}_q^\times$ and T of order 2, respectively. Set $\Gamma = \sum_{x \in \mathbb{F}_q^\times} \psi(x)\lambda(x)$. Let τ range over the characters of $\mathbb{F}_q^\times$ whose squares are nontrivial. Let θ range over the characters of T whose squares are nontrivial. Then the irreducible characters of $G = SL_2(\mathbb{F}_q)$ (for q odd and $q \neq 3$) are given below. Note that $\pi_\tau \simeq \pi_{\tau^{-1}}$ and $\pi_\theta \simeq \pi_{\theta^{-1}}$. In order, the rows give the values of the characters $\chi_{i_B^G \tau}$, the trivial character, the unique irreducible character of degree q, $\chi_{\pi_\lambda^+}$, $\chi_{\pi_\lambda^-}$, χ_{π_θ}, $\chi_{\pi_\nu^+}$, and $\chi_{\pi_\nu^-}$.*

± 1	s_a, $a^2 \neq 1$	$\pm u_1$	$\pm u_\varepsilon$	$\gamma \in T$, $\gamma^2 \neq 1$
$\tau(\pm 1)(q+1)$	$\tau(a) + \tau(a)^{-1}$	$\tau(\pm 1)$	$\tau(\pm 1)$	0
1	1	1	1	1
q	1	0	0	-1
$\lambda(\pm 1)(q+1)/2$	$\lambda(a)$	$\lambda(\pm 1)(1+\Gamma)/2$	$\lambda(\pm 1)(1-\Gamma)/2$	0
$\lambda(\pm 1)(q+1)/2$	$\lambda(a)$	$\lambda(\pm 1)(1-\Gamma)/2$	$\lambda(\pm 1)(1+\Gamma)/2$	0
$\theta(\pm 1)(q-1)$	0	$-\theta(\pm 1)$	$-\theta(\pm 1)$	$-\theta(\gamma) - \theta(\gamma)^{-1}$
$\nu(\pm 1)(q-1)/2$	0	$\nu(\pm 1)(-1+\Gamma)/2$	$\nu(\pm 1)(-1-\Gamma)/2$	$\nu(\gamma)$
$\nu(\pm 1)(q-1)/2$	0	$\nu(\pm 1)(-1-\Gamma)/2$	$\nu(\pm 1)(-1+\Gamma)/2$	$\nu(\gamma)$

In the next chapter we will describe (without proofs) how the above parametrization of the irreducible characters of $SL_2(\mathbb{F}_q)$ fits in with general results on the characters of finite groups of Lie type.