DEPARTMENT OF MATHEMATICS
University of Toronto

Practice Exam in Algebra (3 hours)

1. (a) State the class equation for the finite group G, explaining briefly the meaning of the terms used.

   (b) If G is a finite group of order p^n, where p is a prime and $n \geq 1$, show that the center $Z(G)$ of G is non-trivial.

2. G is a group of order 100. Prove that G is not simple. Must G be solvable? Must G be nilpotent? (Justify your answer.)

3. F is a free abelian group with basis x_1, x_2, x_3, x_4 and H is the subgroup generated by the subset $\{x_1 + x_2 - 2x_3 - x_4,\ 2x_1 - 2x_3,\ 2x_2 + 4x_3 + 4x_4\}$. If $A = F/H$, find the free rank and the invariant factors of A.

4. R is a ring, and M is an R-module.

   (a) If S is a subset of M, explain what it means to say M is a free R-module with basis S.

   (b) If A and B are free R-modules, prove that $A \oplus B$ is a free R-module.

   (c) If $M = A \oplus B$, where A and B are R-modules, and M is free, need A and B be free? (Justify your answer.)

5. R is a commutative ring.

   (a) Define what is meant by a prime ideal of R.

   (b) If J is a prime ideal of R, show that the quotient ring R/J is an integral domain. Is the converse true?

   (c) Give an example of a non-zero prime ideal of $\mathbb{Z}[x]$ which is not a maximal ideal.

6. R is an integral domain, and $N\colon R^{\#} \to \mathbb{Z}^{+}$ is a function (from the non-zero elements of R to the positive integers) such that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in R^{\#}$, and $N(\alpha) = 1$ if and only if α is a unit of R.

   (a) Define what is meant by an irreducible element of R.

(b) If $N(\beta) > 1$, prove that β can be written as a product of (≥ 1) irreducibles.

(c) State an example of such an R which is not a unique factorisation domain.

7. Let p be a prime, and let K be the subfield of $\mathbb{C}$ which is the splitting field over $\mathbb{Q}$ of $x^p - 3$.

   (a) Show $x^p - 3$ is irreducible over $\mathbb{Q}$.

   (b) If α is a primitive pth root of unity in $\mathbb{C}$, explain why $\mathbb{Q}(\alpha)$ is a subfield of K.

   (c) Let G be the Galois group of K over $\mathbb{Q}$. Show that G has order $p(p-1)$, and that G has a normal subgroup H of order p.

8. (a) Find the order of the abelian group

$$(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6) \otimes (\mathbb{Z}_{10} \oplus \mathbb{Z}_{12}) .$$

   (b) Prove that $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = \{0\}$.

9. R is a ring.

   (a) Give a definition of the Jacobson Radical $J(R)$ of R, in terms of the annihilators $A(M)$ of simple R modules M.

   (b) For $y \in R$, show $y \in J(R)$ if and only if $1 - xy$ has a left inverse, for all $x \in R$.

   (c) If F is a field, find $J(F[x_1, \ldots, x_n])$.

N.B. Each ring R is required to have an identity element, and modules and ring homomorphisms are required to be unitary.