DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

January 1995

1. G is a non-abelian group of order 21.

   (a) Prove that G is not simple.

   (b) Describe a composition series for G.

   (c) Must G be solvable? (Justify your answer.)

   (d) Must G be nilpotent? (Justify your answer.)

2. (a) $(A, +)$ is an abelian group, and $S \subset A$. For each abelian M and map $\varphi\colon S \to M$, there is a homomorphism $\Phi\colon A \to M$ whose restriction to S is φ. Does it follow that A is free abelian?

   (b) How many (isomorphism classes of) abelian groups of order 168 are there?

   (c) Prove that $\mathbb{Z}[x, y]/(x^2 - y^2)$ is a Noetherian ring.

   (d) Prove that $\mathbb{Z}_2[x]/(x^3 - 1) \cong Z_2 \times F_4$, where F_4 is a field of order 4.

3. R is a principal ideal domain, and M is a finitely generated R-module.

   (a) Explain briefly the meaning of each of the following:
   (a.1) M has free rank r.
   (a.2) M has invariant factors $d_1, \ldots, d_k$.

   (b) If $M = A_1 \oplus \cdots \oplus A_s$, where the A_i are non-zero cyclic R-modules, show that the number k of invariant factors of M is no more than s.

4. U is a vector space (over the field F) with basis $B = \alpha_1, \ldots, \alpha_r, \beta_{r+1}, \ldots, \beta_n$, and $f : U \to U$ is a linear transformation whose kernel is the span of $\beta_{r+1}, \ldots, \beta_n$.

   (a) Describe a basis of $U \underset{F}{\otimes} U$ in terms of B.

   (b) Describe the map $f \otimes f\colon U \underset{F}{\otimes} U \to U \underset{F}{\otimes} U$.

   (c) Find a basis of the kernel of $f \otimes f$.

5. (a) Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean domain.

   (b) State the factorization of 5 as a product of primes in $\mathbb{Z}[i]$.

   (c) Explain why $F = \mathbb{Z}[i]/(1+2i)$ is a finite field, and find $|F|$.

6. (a) If K is the splitting field over $\mathbb{Q}$ of an irreducible polynomial of degree n, and G is the Galois group of K over $\mathbb{Q}$, explain why G is (isomorphic to) a subgroup of the symmetric group S_n.

   (b) If K is the splitting field of $x^3 - 4x^2 - 6$ over Q, find the Galois group of K over $\mathbb{Q}$.

7. Let R_n $(n = 1, 2, \ldots)$ be rings, and put $S_k = R_1 \times \cdots \times R_k$ and $S_\infty = \prod_{n=1}^{\infty} R_n$. Denote by ε_i the function $R_i \to S_k$ $(1 \leq i \leq k \leq \infty)$ such that $\varepsilon_i(x)$ has ith coordinate x and all other coordinates zero.

   (a) If J is a minimal left ideal of some R_i $(1 \leq i \leq k)$, show that $\varepsilon_i(J)$ is a minimal left ideal of S_k, and that every minimal left ideal of S_k is of this form.

   (b) If each R_i is semisimple, show the S_k are semisimple $(1 \leq k < \infty)$.

   (c) Under what conditions is S_∞ semisimple? (Justify your answer.)

N.B. Each ring R is required to have an identity element, and modules and ring homomorphisms are required to be unitary.