DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

September 1995

Notations: Let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$ denote the fields of the complex, real, and rational numbers, respectively. Let $\mathbb{Z}$ deonte the ring of integers.

1. (a) Prove or disprove that each finite group of order 55 has a normal subgroup of order 11.

   (b) Prove or disprove that each finite group of order 55 has a normal subgroup of order 5.

2. Let R be a ring. Suppose I_i, $i = 1, \ldots, n$, are ideals of R such that $I_i + I_j = R$ for $i \neq j$. Given $r_i \in R$.

   (a) Show there is some $r \in R$ such that $r \equiv r_i \mod I_i$ for all i.

   (b) Show, if $r' \in R$ and $r' \equiv r_i \mod I_i$ for all i, then $r' \equiv r \mod \bigcap_{i=1}^{n} I_i$.

   (c) Show there is an isomorphism

   $$f : \frac{R}{I_1 \cap \cdots \cap I_n} \to \frac{R}{I_1} \oplus \cdots \oplus \frac{R}{I_n}.$$

   (d) Let $m_i \in \mathbb{Z}$, $i = 1, \ldots, n$, such that $(m_i, m_j) = 1$ for $i \neq j$. Show $\mathbb{Z}_{m_1 \ldots m_n} \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_n}$, ($\mathbb{Z}_m = \mathbb{Z}/(m)$).

   (e) Show $\mathbb{C}[x]/(x^n - 1) \cong \mathbb{C}[x]/(x - 1) \oplus \cdots \oplus \mathbb{C}[x]/(x - \zeta^{n-1})$.

3. Suppose R is a ring and N is a unitary left R-module (i.e. $1 \cdot n = n$ for all $n \in N$). Show, $R \otimes_R N$ is R-isomorphic with N.

4. Let G be a finite group and F a field. Let FG denote the group ring (group algebra) of G over F.

   (a) Give a criterion for FG to be semisimple.

   (b) In (i)–(v) write FG as a direct sum of indecomposable ideals of FG if possible. (An ideal is indecomposable if it is not the direct sum of two nonzero subideals.)

      (i) $F = \mathbb{C}$ and G is commutative

      (ii) $F = \mathbb{C}$ and $G = S_3$ (the symmetric group of three elements)

      (iii) $F = \mathbb{C}$ and $G = C_3$ (the cyclic group of order 3)

      (iv) $F = \mathbb{R}$ and $G = C_3$

      (v) $F = \mathbb{Q}$ and $G = C_3$

5. Let V be a unitary finite-dimensional vector space over $\mathbb{C}$.

   (a) Suppose T is a self-adjoint linear transformation on V. Show that (Ty, x), $x, y \in V$, is a Hermitian inner product. We say T is positive if $(Tx, x) \geq 0$, and T is strictly positive if T is positive and $(Tx, x) = 0$ implies $x = 0$.

   (b) If T is any linear transformation on a unitary space V, then T^* denotes the adjoint transformation of T. Is T^*T self-adjoint? Is T^*T positive? Is T^*T strictly positive if T is nonsingular?

   (c) Let T be a positive (and therefore self-adjoint) linear transformation on a unitary space V. Show there is a unique positive transformation H on V such that $H^2 = T$. Show also that H commutes with every transformation that commutes with T.

   (d) Let T be a nonsingular linear transformation on a unitary space V. Show that T can be written $T = UH$, where H is positive and U is unitary; both, H and U, are unique.