DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

January 1996

1. Let G be a finite group with subgroups H and K.
   (a) Show that there is a double coset decomposition
   $$G = \bigcup H a_i K \ , \qquad a_i \in G \ ,$$
   into disjoint sets.
   (b) Is $|Ha_iK| = |Ha_jK|$ for all a_i, a_j? If not, give a specific counterexample.
   (c) Show, for any $a \in G$ we have
   $$|HaK| \cdot |a^{-1}Ha \cap K| = |H| \cdot |K| \ .$$

2. Let G be a group of order pqr, where p, q, r are distinct primes. Show that G is not simple.

3. Let R be a ring, M a right R-module, and N a left R-module.
   (a) Define the tensor product $M \otimes_R N$.
   (b) Let $R = \mathbb{Z}$, $M = \mathbb{Q}$, $N = \mathbb{Z}_n = \mathbb{Z}/(n)$. What is $M \otimes_{\mathbb{Z}} \mathbb{Z}_n$?
   (c) Let M_1, M_2 be right and let N_1, N_2 be left R-modules. Suppose $f \in \mathrm{Hom}_R(M_1, M_2)$ and $g \in \mathrm{Hom}_R(N_1, N_2)$. Show there is a unique $h \in \mathrm{Hom}_{\mathbb{Z}}(M_1 \otimes_R N_1,\ M_2 \otimes_R N_2)$ such that $h(x \otimes y) = f(x) \otimes g(y)$ for all $x \in M_1$, $y \in N_1$.
   (d) Let S be a ring. Assume M is an $S-R$ bimodule. Show, $M \otimes_R N$ can be made into an S-module.

4. Let F and K be fields such that $F \subset K$. Define when a set $S \subset K$ is algebraically independent over F. Show there is a transcendence basis B for K over F. Show in particular that K is algebraic over $F(B)$.

5. Give four characterizations of a semisimple ring. Choose two of them and show they are equivalent.

6. Show that every finite subgroup of the multiplicative group of a field is cyclic. Is this statement still true if we replace field by skewfield?

7. Let R be a simple ring with unit element. Suppose that R has a minimal right ideal. Then R is an artinian ring.