

DEPARTMENT OF MATHEMATICS  
University of Toronto

**Algebra Exam (3 hours)**

*September 1996*

1. Suppose that  $G$  is a group.
  - (a) Define  $G$ -set (i.e.  $G$ -action on a set), transitive  $G$ -set,  $G$ -isomorphism.
  - (b) Suppose  $S$  is a transitive  $G$ -set. Show that  $S$  is  $G$ -isomorphic to  $G/K$ , for a subgroup  $K$  of  $G$ .
  - (c) If  $K_1$  and  $K_2$  are subgroups of  $G$ , give necessary and sufficient conditions that the transitive  $G$ -sets  $G/K_1$  and  $G/K_2$  be  $G$ -isomorphic.
2.
  - (a) How many abelian groups are there, up to isomorphism, of order 100,000?
  - (b) One of these groups is a direct product of cyclic groups of prime order. For this group, identify the invariant factors and the elementary divisors.
3. Suppose that  $K/F$  is a Galois extension of fields, with  $\text{Gal}(K/F) = \text{GL}(n, \mathbb{F}_p)$ ,  $p$  prime.
  - (a) What is the dimension of  $K$  as a vector space over  $F$ ?

Let  $K_1, K'_1$  and  $K_2$  be the subfields of  $K$  which contain  $F$  and which correspond to subgroups  $\text{Gal}(K/K_1) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ ,  $\text{Gal}(K/K'_1) = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$  and  $\text{Gal}(K/K_2) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$  of  $\text{GL}(2, \mathbb{F}_p)$ .
  - (b) Prove that  $K_2/K_1$  is a Galois extension, and compute  $\text{Gal}(K_2/K_1)$ .
  - (c) Compute the Galois group of  $K$  over the composite field  $K_1 K'_1$ .
  - (d) Prove that  $K'_1 \cap K_2 = \mathbb{Q}$ .
4.
  - (a) Define Noetherian ring, integral domain, principal ideal domain, unique factorization domain, Euclidean domain.

- (b) Describe all the relationships among these six properties (i.e. which properties imply which other properties).
  - (c) If  $R$  is a ring which satisfies one of the six properties, does  $R[x]$  have the same property? In each case, state the appropriate theorem or give a counterexample.
5. A module  $M \neq \{0\}$  over a ring  $R$  is *simple* if the only submodules are  $\{0\}$  and  $M$ .
- (a) Show that  $M$  is simple if and only if it is cyclic, with any nonzero element being a generator.
  - (b) If  $M_1$  and  $M_2$  are simple, show that any  $R$ -homomorphism from  $M_1$  to  $M_2$  is an  $R$ -isomorphism.
  - (c) If  $M$  is simple, show that  $\text{End}_R(M)$  is a division ring.
6. A matrix  $A \in M_n(F)$ ,  $F$  a field, is *semisimple* if  $F^n$  is a direct sum of simple  $F[x]$ -submodules. (Recall that  $A$  makes  $F^n$  into an  $F[x]$ -module.)
- (a) Suppose that  $F = \mathbb{C}$ . Show that  $A$  is semisimple if and only if it is diagonalizable.
  - (b) Show by counterexample that (a) is false for arbitrary  $F$ .
  - (c) Show that any matrix  $A \in M_n(\mathbb{C})$  can be written  $A = S + N$ , where  $S$  is semisimple,  $N$  is nilpotent and  $SN = NS$ .