DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

September 1996

1. Suppose that G is a group.

   (a) Define G-set (i.e. G-action on a set), transitive G-set, G-isomorphism.

   (b) Suppose S is a transitive G-set. Show that S is G-isomorphic to G/K, for a subgroup K of G.

   (c) If K_1 and K_2 are subgroups of G, give necessary and sufficient conditions that the transitive G-sets G/K_1 and G/K_2 be G-isomorphic.

2. (a) How many abelian groups are there, up to isomorphism, of order 100,000?

   (b) One of these groups is a direct product of cyclic groups of prime order. For this group, identify the invariant factors and the elementary divisors.

3. Suppose that K/F is a Galois extension of fields, with $\mathrm{Gal}(K/F) = \mathrm{GL}(n, \mathbb{F}_p)$, p prime.

   (a) What is the dimension of K as a vector space over F?

   Let K_1, K_1' and K_2 be the subfields of K which contain F and which correspond to subgroups $\mathrm{Gal}(K/K_1) = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$, $\mathrm{Gal}(K/K_1') = \left\{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}\right\}$ and $\mathrm{Gal}(K/K_2) = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$ of $\mathrm{GL}(2, \mathbb{F}_p)$.

   (b) Prove that K_2/K_1 is a Galois extension, and compute $\mathrm{Gal}(K_2/K_1)$.

   (c) Compute the Galois group of K over the composite field K_1K_1'.

   (d) Prove that $K_1' \cap K_2 = \mathbb{Q}$.

4. (a) Define Noetherian ring, integral domain, principal ideal domain, unique factorization domain, Euclidean domain.

(b) Describe all the relationships among these six properties (i.e. which properties imply which other properties).

(c) If R is a ring which satisfies one of the six properties, does $R[x]$ have the same property? In each case, state the appropriate theorem or give a counterexample.

5. A module $M \neq \{0\}$ over a ring R is *simple* if the only submodules are $\{0\}$ and M.

   (a) Show that M is simple if and only if it is cyclic, with any nonzero element being a generator.

   (b) If M_1 and M_2 are simple, show that any R-homomorphism from M_1 to M_2 is an R-isomorphism.

   (c) If M is simple, show that $\mathrm{End}_R(M)$ is a division ring.

6. A matrix $A \in M_n(F)$, F a field, is *semisimple* if F^n is a direct sum of simple $F[x]$-submodules. (Recall that A makes F^n into an $F[x]$-module.)

   (a) Suppose that $F = \mathbb{C}$. Show that A is semisimple if and only if it is diagonalizable.

   (b) Show by counterexample that (a) is false for arbitrary F.

   (c) Show that any matrix $A \in M_n(\mathbb{C})$ can be written $A = S + N$, where S is semisimple, N is nilpotent and $SN = NS$.