DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

January 1997

No aids.
Do all questions.

1. *[20 points]*
   (a) Prove that the following conditions on a ring R are equivalent.
      (i) R satisfies the ascending chain condition for left ideals.
      (ii) Any nonempty set $\mathcal{S}$ of left ideals has a maximal element.
      (iii) Any left ideal in R is finitely generated.
   (b) Suppose that R satisfies the three conditions of (a), and that I is a left ideal in $R[x]$. Show that any $R[x]$-submodule of $R[x]/I$ is finitely generated.

2. *[15 points]*
   (a) Prove that any element $g \in GL(n, \mathbb{C})$ has an eigenvalue.
   (b) Let $G_{1,2}$ be the subset of $G = GL(4, \mathbb{C})$ whose eigenvalues are in the set $\{1, 2\}$. Prove that $G_{1,2}$ is invariant under conjugation by G.
   (c) How many G-conjugacy classes are there in $G_{1,2}$?

3. *[15 points]*
   Suppose that R is an integral domain.
   (a) Define a *prime* element and an *irreducible* element in R.
   (b) Prove that any prime element is irreducible.
   (c) Prove that any irreducible element $f(x) \in \mathbb{Z}[x]$ is prime.
   (d) Write down an irreducible polynomial in $\mathbb{Z}[x]$ of degree 6. (Explain your reasons.)

4. *[30 points]*
   a) State Sylow's theorem.
   (b) What is the order of the group

$$SL(2, \mathbb{F}_3) \;=\; \{g \in M_2(\mathbb{F}_3) :\; \det(g) = 1\}\ ,$$

   and what is the order of its center Z?
   (c) Write down a Sylow 3-subgroup of $SL(2, \mathbb{F}_3)$.
   (d) How many Sylow 3-subgroups does $SL(2, \mathbb{F}_3)$ have?
   (e) Show that $SL(2, \mathbb{F}_3)/Z \cong A_4$.

5. *[20 points]*
   Let $f(x) \in F[x]$ be a monic polynomial of degree n.
   (a) What is meant by the splitting field E of $f(x)$ over F?
   (b) Define the Galois group of E/F, <u>and</u> the Galois group of $f(x)$ over F, and describe how the two are related.
   (c) If $\{\alpha_1, \ldots, \alpha_n\}$ are the roots of $f(x)$, show that

$$D \;=\; \prod_{1 \le i,\, j \le n} (\alpha_i - \alpha_j)^2$$

   belongs to F.
   (d) Show that the Galois group of $f(x)$ over F is contained in A_n if and only if D has a square root in F.