DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

May 1997

No aids.
Do all questions.

1. *[15 points]*
   (a) Define solvable, nilpotent, p-group and Sylow p-subgroup, for a finite group G.
   (b) State Sylow's theorem.
   (c) Prove that if $|G| = 105$, then G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.

2. *[20 points]*
   Let S_5 act on $(\mathbb{F}_5)^5$ by permuting the co-ordinates, and form the semi-direct product

$$G = (\mathbb{F}_5)^5 \rtimes S_5.$$

   (a) What is the order of G?
   (b) How many Sylow 5-subgroups does G have; write down one of them.
   (c) Determine the composition factors of G (up to isomorphism).

3. *[25 points]*
   (a) Define Euclidean domain, principal ideal domain, unique factorization domain.
   (b) Prove that a Euclidean domain is a principal ideal domain.
   (c) Prove that a principal ideal domain is a unique factorization domain.
   (d) Let $R = \{m + n\sqrt{-1}\colon\ m, n \in \mathbb{Z}\}$. Prove that R is a Euclidean domain. (Hint: If $a, b \in R$, with $b \neq 0$, find an element $q \in R$ with $\left|\frac{a}{b} - q\right| \leq \frac{1}{\sqrt{2}}$.)

4. *[20 points]*

a) Let T be the linear transformation on the complex vector space

$$\mathbb{C}[x]/\big((x+1)^2\big) \oplus \mathbb{C}[x]/\big((x-1)(x^2+1)^2(x^4-1)\big) \oplus \mathbb{C}[x]/\big((x+1)(x^2-1)\big)$$

obtained by multiplying by x. What are the invariant factors and elementary divisors of T?

(b) For the matrix

$$A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix},$$

compute the invariant factors, elementary divisors, minimal polynomial, characteristic polynomial, rational canonical form and Jordan canonical form.

5. *[20 points]*

(a) State in full generality the fundamental theorem of Galois theory for finite Galois extensions K/F. Include as many direct corollaries of the theorem as you can.

(b) Let K be the splitting field of $f(x) = x^4 - 2$ over $\mathbb{Q}$, and compute $G = \mathrm{Gal}(K/\mathbb{Q})$.

(c) Find all fields E, $\mathbb{Q} \subset E \subset K$, and identify the corresponding subgroups H of G.