DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

September 1997

No aids.
Do all questions.
Each of the five questions is worth 20 points.

1. Let p and q be distinct primes with $p < q$. Prove that the number of non-isomorphic groups of order pq is
$$\begin{cases} 2 & \text{if } q \equiv 1 \text{mod } p \\ 1 & \text{otherwise.} \end{cases}$$

2. Let R be a commutative ring with identity. Define the terms prime ideal and maximal ideal. Prove that R has a unique prime ideal if and only if every non-unit is nilpotent.

3. (a) Let p be a prime. Prove that
$$X^{p-1} + X^{p-2} + \cdots + X + 1$$
is an irreducible polynomial in $\mathbb{Z}[X]$.

   (b) Compute the Galois group of the (splitting fields of the) polynomials $X^3 - 2$ and $X^3 - 4X + 2$ over $\mathbb{Q}$.

4. (a) Let A be an $n \times n$ matrix over the complex numbers. Prove that A is nilpotent if and only if the trace of A^r is zero for all $r \geq 1$.

   (b) Find all possible rational canonical forms for a 6×6 matrix with rational entries and having minimal polynomial $(X-2)^2(X+3)$. Find all possible Jordan canonical forms for a 6×6 matrix with complex entries and having the same minimal polynomial.

5. Let R be a ring with identity.

   (a) Give a definition of the Jacobson radical $J(R)$ of R.

   (b) For $y \in R$ show that $y \in J(R)$ if and only if $1 - xy$ has a left inverse for all $x \in R$.

   (c) If F is a field, find $J(F[X_1, ..., X_n])$.