DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

September 1998

No aids.
Do all questions.
All questions are of equal value.

1. Suppose A is an 8×8 matrix whose characteristic polynomial is $\lambda^3(\lambda-1)^2(\lambda-5)^3$ and whose minimal polynomial is $\lambda(\lambda-1)^2(\lambda-5)^2$. What is the dimension of each of its eigenspaces?

2. Classify all groups of order 12.

3. What is the Galois group of (the splitting field of) $X^3 - 2$ over $\mathbb{Q}$? (Explain your answer.)

4. Give a set of representatives for the conjugacy classes in the symmetric group S_5, and find the number of elements in each class.

5. If p is a prime, what is the order of $GL(2, \mathbb{F}_p)$, the group of invertible 2×2 matrices over the field with p elements?

6. For which primes p is every element of the finite field $\mathbb{F}_p$ equal to the fifth power of an element of the field?

7. Suppose $f \in \mathbb{Q}[x]$ is an irreducible cubic polynomial for which the Galois group of the splitting field of f over $\mathbb{Q}$ is not abelian. How many subfields does the splitting field of f have, and how many of them are normal? (Explain your answer.)

8. (a) What are the maximal ideals in $\mathbb{C}[x]$? Explain your answer.

(b) Give an example of a ring that contains a prime ideal that is not maximal. Explain your answer.

9. What are the maximal ideals in $\mathbb{R}[x]$? Explain your answer. (Hint: What is

$$\mathbb{R}[x]/(x^2+1),$$

where (x^2+1) means the principal ideal generated by x^2+1?)

10. Both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are modules over the ring $\mathbb{Z}_6$ in a natural way. Identify the tensor product

$$\mathbb{Z}_2 \otimes_{\mathbb{Z}_6} \mathbb{Z}_3.$$

(Here $\mathbb{Z}_m$ means $\mathbb{Z}/(m\mathbb{Z})$....)