DEPARTMENT OF MATHEMATICS

University of Toronto

Algebra Exam (3 hours)

September 9, 1999

No aids.
Do all questions.
Each of the five questions is worth 20 marks.

1. (a) If T is a diagonalizable linear operator on a vector space V of finite dimension, and if the characteristic polynomial of T has only one root, show that T is a scalar multiple of the identity.

(b) Let A be an $n \times n$ real matrix with n real distinct eigenvalues. If B is a real $n \times n$ matrix which commutes with A, show that B is diagonalizable.

(c) Let M be a real $n \times n$ symmetric matrix. Prove that M can be diagonalized and that all its eigenvalues are real.

(d) Compute the minimal polynomial of the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and write down its Jordan canonical form.

2. Let G be a group of order pqr where p, q and r are distinct primes. Show that G is not simple. Show that G is solvable.

3. Let R be a commutative ring with identity and J an ideal which is contained in every maximal ideal of R. Let M be a finitely generated R-module such that $JM = M$. Prove that $M = 0$.

4. (a) Consider the field $K = \mathbb{Q}(2^{1/5})$. Determine the Galois closure L of $K/\mathbb{Q}$ and the Galois group of $\mathrm{Gal}(L/\mathbb{Q})$.

(b) Let K be a field containing the rational numbers and the n-th roots of unity. If L/K is a cyclic extension (that is, a Galois extension with cyclic Galois group) of degree n, then prove that there is an element α in K such that $L = K(\alpha^{1/n})$.

5. Let p be a prime and F a finite field of p elements. Let $\overline{F}$ be an algebraic closure of F.

(a) Show that for each integer $N \geq 1$, there is a unique extension F_N of F contained in $\overline{F}$ of degree N.

(b) Show that F_N/F is Galois with group $\mathbb{Z}/N\mathbb{Z}$.

(c) Compute $\mathrm{Gal}(\overline{F}/F)$.