**DEPARTMENT OF MATHEMATICS**
**University of Toronto**

# Algebra Exam
**May 2000**

Time: 3 hours | No aids allowed

1. *[20 points]*
   a) Define solvable, nilpotent, $p$-group, and Sylow $p$-subgroup, for a finite group.
   b) State Sylow's theorem.
   c) Let $G$ be a group of order 231. Show that $G$ has a normal Sylow 7-subgroup, and there is a Sylow 11-subgroup of $G$ contained in the centre of $G$.
   d) Let $G$ be a nonabelian group of order 231. Is $G$ nilpotent? Is $G$ solvable? (Justify your answers).

2. *[20 points]*
   Let $T$ be the linear transformation on the complex vector space

$$V = \mathbb{C}[x]/(x-2) \oplus \mathbb{C}[x]/(x^2+1)(x-2)^2 \oplus \mathbb{C}[x]/(x^2+1)(x^2-4)$$

   obtained by multiplying by $x$.
   a) Find the invariant factors and elementary divisors of $T$.
   b) Determine the rational canonical form and the Jordan canonical form of $T$.
   c) Find the minimal and characteristic polynomials of $T$.

3. *[20 points]*
   Let $R$ be a commutative ring with 1.
   a) Prove that if $R$ is finite, then every prime ideal of $R$ is a maximal ideal.
   b) Prove that if $R$ is a principal ideal domain, then every prime ideal of $R$ is a maximal ideal.
   c) Define unique factorization domain, Euclidean domain, Noetherian ring.
   d) In each of the following cases,
   (i) $R$ a unique factorization domain
   (ii) $R$ a Euclidean domain
   (iii) $R$ Noetherian

   is a prime ideal of $R$ necessarily maximal? If yes, give reasons. If no, give a counterexample.

4. *[20 points]*

   Let $R$ be a ring with 1.

   a) Define cyclic left $R$-module and simple left $R$-module.

   b) Prove that a simple left $R$-module is cyclic. Is a cyclic left $R$-module simple?

   c) A left $R$-module $M$ is said to be *faithful* if $\{ r \in R \mid rM = 0 \} = 0$. Prove that there exists a faithful simple left $R$-module if and only if there exists a maximal left ideal $I$ of $R$ such that $\{0\}$ and $I$ are the only left ideals of $R$ contained in $I$.

5. *[20 points]*

   Let $\mathbb{F}_p$ be the finite field with $p$ elements, $p$ prime. Suppose that $L/K$ is a Galois extension of fields such that $\mathrm{Gal}(L/K) = GL_2(\mathbb{F}_p)$. Let $L_1$ and $L_2$ be the subfields of $L$ containing $K$ which correspond to the subgroups

$$\mathrm{Gal}(L/L_1) = SL_2(\mathbb{F}_p) \quad \text{and} \quad \mathrm{Gal}(L/L_2) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in \mathbb{F}_p^\times,\ b \in \mathbb{F}_p \right\}$$

   of $\mathrm{Gal}(L/K)$.

   a) Compute the degrees $[L_1 : K]$, $[L_2 : K]$, and $[L_1 \cap L_2 : K]$.

   b) Let $L_1 L_2$ be the composite of $L_1$ and $L_2$. Prove that $L_1 L_2 / L_2$ is a Galois extension, but $L_1 L_2 / K$ is not a Galois extension.

   c) Compute $\mathrm{Gal}(L/L_1 L_2)$ and $\mathrm{Gal}(L_1 L_2 / L_2)$.