DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam
May 2000

Time: 3 hours | No aids allowed

1. *[20 points]*
   a) Define solvable, nilpotent, p-group, and Sylow p-subgroup, for a finite group.
   b) State Sylow's theorem.
   c) Let G be a group of order 231. Show that G has a normal Sylow 7-subgroup, and there is a Sylow 11-subgroup of G contained in the centre of G.
   d) Let G be a nonabelian group of order 231. Is G nilpotent? Is G solvable? (Justify your answers).

2. *[20 points]*
   Let T be the linear transformation on the complex vector space

$$V = \mathbb{C}[x]/(x-2) \oplus \mathbb{C}[x]/(x^2+1)(x-2)^2 \oplus \mathbb{C}[x]/(x^2+1)(x^2-4)$$

   obtained by multiplying by x.
   a) Find the invariant factors and elementary divisors of T.
   b) Determine the rational canonical form and the Jordan canonical form of T.
   c) Find the minimal and characteristic polynomials of T.

3. *[20 points]*
   Let R be a commutative ring with 1.
   a) Prove that if R is finite, then every prime ideal of R is a maximal ideal.
   b) Prove that if R is a principal ideal domain, then every prime ideal of R is a maximal ideal.
   c) Define unique factorization domain, Euclidean domain, Noetherian ring.
   d) In each of the following cases,
   (i) R a unique factorization domain
   (ii) R a Euclidean domain
   (iii) R Noetherian

   is a prime ideal of R necessarily maximal? If yes, give reasons. If no, give a counterexample.

4. *[20 points]*
   Let R be a ring with 1.
   a) Define cyclic left R-module and simple left R-module.
   b) Prove that a simple left R-module is cyclic. Is a cyclic left R-module simple?
   c) A left R-module M is said to be *faithful* if $\{ r \in R \mid rM = 0 \} = 0$. Prove that there exists a faithful simple left R-module if and only if there exists a maximal left ideal I of R such that $\{0\}$ and I are the only left ideals of R contained in I.

5. *[20 points]*
   Let $\mathbb{F}_p$ be the finite field with p elements, p prime. Suppose that L/K is a Galois extension of fields such that $\mathrm{Gal}(L/K) = GL_2(\mathbb{F}_p)$. Let L_1 and L_2 be the subfields of L containing K which correspond to the subgroups

$$\mathrm{Gal}(L/L_1) = SL_2(\mathbb{F}_p) \quad \text{and} \quad \mathrm{Gal}(L/L_2) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in \mathbb{F}_p^\times,\ b \in \mathbb{F}_p \right\}$$

   of $\mathrm{Gal}(L/K)$.
   a) Compute the degrees $[L_1 : K]$, $[L_2 : K]$, and $[L_1 \cap L_2 : K]$.
   b) Let L_1L_2 be the composite of L_1 and L_2. Prove that L_1L_2/L_2 is a Galois extension, but L_1L_2/K is not a Galois extension.
   c) Compute $\mathrm{Gal}(L/L_1L_2)$ and $\mathrm{Gal}(L_1L_2/L_2)$.