DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam
September 7, 2000

Time: 3 hours No aids allowed

1. *[20 points]*
   Let S_n be the symmetric group on n letters, $n \geq 2$.
   a) Find the cycle decomposition of the element $(1\,8\,10\,4)(3\,4\,9\,8)$ of S_{10}. What is its order?
   b) Let $\sigma = (4\,2\,1)(3\,6\,7\,5)$ and $\tau = (4\,5)(7\,6\,1)$. Compute $\tau\sigma\tau^{-1}$.
   c) Describe the conjugacy class of the element $(12\cdots n)$ in S_n.
   d) Let $H = \langle(12\cdots n)\rangle$ be the subgroup of S_n generated by $(12\cdots n)$. Find the normalizer of N of H in S_n.

2. *[20 points]*
   Let T be the linear transformation on the complex vector space

$$V = \mathbb{C}[x]/(x^2-1)(x+1) \oplus \mathbb{C}[x]/(x^4-1) \oplus \mathbb{C}[x]/(x^2+1)^2(x-1)$$

   obtained by multiplying by x.
   a) Find the invariant factors and elementary divisors of T.
   b) Determine the rational canonical form and the Jordan canonical form of T.
   c) Find the minimal and characteristic polynomials of T.

3. *[20 points]*
   Let R be an integral domain.
   a) Define irreducible element of R, prime element of R, and associate elements in R.
   b) Prove that a prime element of R is irreducible.
   c) Prove that an element $r \in R$ is irreducible if and only if the ideal (r) is maximal among the set of proper principal ideals in R. (Here the set of principal ideals is ordered by inclusion).
   d) Define UFD (unique factorization domain).
   e) Prove that an irreducible element in a UFD is prime.

R is said to satisfy the ascending chain condition for principal ideals, abbreviated ACCPI, if whenever $I_1 \subset I_2 \subset \cdots$ is an increasing sequence of principal ideals in R, there exists an n such that $I_m = I_n$ for all $m \geq n$.

f) Prove that R is a UFD if and only if R satisfies the ACCPI and every irreducible element of R is prime.

4. *[20 points]*

Let R be a ring with 1.

a) Define cyclic left R-module and simple left R-module. (*Note*: irreducible is the same as simple).
b) Prove that a nonzero left R-module M is simple if and only M is a cyclic left R-module with any nonzero element as generator.
c) Suppose that M_1 and M_2 are simple left R-modules. Show that any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Prove that $\mathrm{End}_R(M_1)$ is a division ring.

5. *[20 points]*

a) Define field, field extension, separable field extension, finite Galois extension, and Galois group of a finite Galois extension.
b) State the fundamental theorem of Galois theory.

Let M be a finite Galois extension of a field K, with Galois group $\mathrm{Gal}(M/K)$. Suppose that L is a subfield of M containing K. Set $G = \{\, g \in \mathrm{Gal}(M/K) \mid \sigma(L) \subset L \,\}$.

c) Prove that G is the normalizer of $\mathrm{Gal}(M/L)$ in $\mathrm{Gal}(M/K)$.
d) Let $\mathrm{Aut}(L/K)$ be the set of automorphisms of L which fix K pointwise. Prove that $G/\mathrm{Gal}(M/L) \simeq \mathrm{Aut}(L/K)$.