DEPARTMENT OF MATHEMATICS

University of Toronto

Algebra Exam

September 6, 2001

Time: 3 hours — No aids allowed

1. Let G be a group of order 56 and let $f,\ h : G \to G$ be maps such that

$$f(x) = x^3, \qquad h(x) = x^4.$$

a) Show that G is abelian $\iff$ f, g are homomorphisms.
b) State a generalization of **a**) for an arbitrary group of order $u \geq 3$.
c) Use **b**) to show that a non-cyclic group of order 4 is abelian.

2. Let A be a commutative ring with 1 and let $P = (x)$ be a principal ideal of A. Consider $I = \bigcap_{n=1}^{\infty} P^n$.

a) Suppose P is prime. Let Q be a prime ideal of A such that $Q \subsetneq P$. Show that $Q \subset I$.
b) Assume that x is not a zero divisor in A. Show that I is prime and $I = xI$.
c) Assume that A is an integral domain, that P is prime, and that I is finitely generated. Prove that $I = (0)$.

3. Let A be a $n \times n$ matrix with entries in $\mathbb{C}$ (*i.e.* $A \in \mathrm{M}_n(\mathbb{C})$), such that $A^r = I$ for some $r \in \mathbb{N}$.

a) Show that if A has a unique eigenvalue ζ, then $A = \zeta I$
b) Assume that $A \in \mathrm{M}_n(\mathbb{F}_2)$, where $\mathbb{F}_2$ is a finite field with 2 elements. Does **a**) still hold? If not, produce a counterexample.
c) Let k be a field. Prove that the ring $\mathrm{M}_n(k)$ contains an isomorphic copy of every extension of k of degree at most n.

4. Let K be a field and let $F = K(a)$, $L = K(b)$ be two extensions of K (both contained in an algebraic closure $\overline{K}$ of K).

a) Assume that F and L are normal, separable extensions of K and that the extension degrees $[F : K]$ and $[L : K]$ are coprime. Show that $a+b$ generates the composite extension FL.
b) Assume only that $F \cap L = K$. Give an example where $a + b$ does not generate FL.

5. Let $K/\mathbb{Q}$ be the splitting field of the cyclotomic polynomial $\phi_{10}(x)$.

a) Describe K and determine the degree of the extension $[K : \mathbb{Q}]$.

b) Determine the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ as well as the complete relationship between subgroups of G and subfields of K. Does G contain any subgroup of order 5?