DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

May 8, 2002

No aids.
Do all questions.

1. Let G be a finite abelian group, and let $\widehat{G}$ be the set of all homomorphisms from G to $\mathbb{T} = \{z \in \mathbb{C} | |z| = 1\}$. Then $\widehat{G}$ is an abelian group under the group operation of pointwise multiplication of functions. Let $\widehat{\widehat{G}}$ be the group of homomorphisms from $\widehat{G}$ to $\mathbb{T}$. Give an outline of a proof of the following:

(a) $G \simeq \widehat{G}$ (not canonically)

(b) The map $\phi : G \longrightarrow \widehat{\widehat{G}}$, $g \mapsto \phi(g)$, where $\phi(g)(\chi) = \chi(g)$ for all $\chi \in \widehat{G}$, is an isomorphism.

(c) $\sum_{g \in G} \chi(g) = \begin{cases} |G|, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 \end{cases}$

(d) $\sum_{\chi \in \widehat{G}} \chi(g) = \begin{cases} |G|, & \text{if } g = e \\ 0, & \text{if } g \neq e \end{cases}$ (e is the identity in G)

2. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

3. Let R be the ring of all continuous functions from the closed interval [0,1] to $\mathbb{R}$ and for each $c \in [0, 1]$, let $M_c = \{f \in R | f(c) = 0\}$. Since $R/M_c \simeq \mathbb{R}$, M_c is a maximal ideal. Show

(a) If M is any maximal ideal of R, then there exists a real number $c \in [0, 1]$ such that $M = M_c$.

(b) M_c is not equal to the principal ideal generated by $x - c$.

4. Let K be a field of characteristic zero. Let G be the subgroup of $Aut\big(K(x)/K\big)$ that is generated by the autmorphism $\phi : x \mapsto x + 1$. Determine the fixed field E of G.

5. Let G be a simple group of order 60. Suppose G has a subgroup H of order 12. Prove $G \simeq A_5$ (the alternating group on 5 letters).

6. Let F be a field such that no polynomial of odd degree is irreducible over F. Let E/F be a finite Galois extension. Prove that $[E : F] = 2^n$ for some positive integer n.