DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

Thursday, September 5, 2002, 1-4 p.m.

No aids.
Do all questions.

1. Let R be a commutative ring with unity. Recall the definition of a radical of an ideal: Let I be a (proper) ideal of R. The radical of I is defined by

$$\sqrt{I} = \{r \in R \mid \quad \text{there exists } n \in \mathbb{N} \text{ such that } r^n \in I\}.$$

Prove that any prime ideal P satisfies $P = \sqrt{P}$. More generally, prove that for any ideal I, $\sqrt{I}$ is the intersection of all prime ideals containing I.

2. Prove that any commuting set of diagonalizable linear transformations is simultaneously diagonalizable. More precisely, let V be a finite dimensional vector space over a field k and let $\{T_\alpha\}_{\alpha \in A}$ be a set of commuting linear transformations on V which are diagonalizable. Then there exists a basis of V such that the matrices of T_α with respect to that basis are all diagonal matrices for all $\alpha \in A$.

3. Let G be the subgroup of $Aut(\mathbb{F}_p(x)/\mathbb{F}_p)$ that is generated by the automorphism $\phi : x \longmapsto x + 1$, where x is an indeterminate. Determine the fixed field E of G.

4. Let G be a p-group, namely, the order of G is p^k for some positive integer k, where p is a prime. Prove that the center of G is non-trivial. Using this, prove that any p-group is solvable.

5. Let E/F be a finite extension of finite fields. Prove that the norm map $N_{E/F} : E^\times \longrightarrow F^\times$ is surjective.