DEPARTMENT OF MATHEMATICS
University of Toronto

# Algebra Exam (3 hours)

*September 2003*

No aids allowed.
Do all questions.

**1.** *[20 points]*

a) Define nilpotent for a finite group.

b) Let $D_{12} = \langle r, s \mid r^6 = s^2 = 1, \ rs = sr^{-1} \rangle$ be the dihedral group of order 12. Determine whether $D_{12}$ is a nilpotent group.

**2.** *[20 points]*

Let $R$ be a commutative ring with identity.

a) Prove that if $R$ is finite, then every nonzero prime ideal of $R$ is a maximal ideal.

b) An ideal $I$ of $R$ is primary if whenever $rs \in I$ and $r \notin I$, then $s^n \in I$ for some positive integer $n$. Prove that $I$ is primary if and only if every zero divisor in $R/I$ is a nilpotent element.

**3.** *[20 points]*

Let $R$ be a PID, and let $M$ be a finitely generated $R$-module of free rank (aka Betti number) 0.

a) State the fundamental theorem giving the invariant factor decomposition of $M$.

b) Prove that $M$ is a cyclic $R$-module if and only if $M$ has exactly one invariant factor.

c) Prove that $M$ is a simple (aka irreducible) $R$-module if and only if $M$ has exactly one invariant factor, and the invariant factor is prime.

**4.** *[15 points]*
Let $F$ be a field and let $f(x) \in F[x]$. Prove that there exists an extension $E$ of $F$ which contains a root of $f(x)$.

**5.** *[25 points]*
Let $E$ be a finite Galois extension of a field $F$, with Galois group $G = \mathrm{Gal}(E/F)$.

a) Let $n = [E : F]$. Let $m$ be a positive integer dividing $n$, and set $\mathcal{F}_m = \{ K \text{ a field} \mid F \subset K \subset E,\ [K : F] = m \}$. Suppose that $\mathcal{F}_m$ is nonempty. Show that the map $G \times \mathcal{F}_m \to \mathcal{F}_m$ given by $(\sigma, K) \mapsto \sigma(K)$, $\sigma \in G$, $K \in \mathcal{F}_m$, defines an action of $G$ on the set $\mathcal{F}_m$. Prove that this action is transitive if and only if any two subgroups of index $m$ in $G$ are conjugate in $G$.

b) Assume that $G = A_5$ (the alternating group on 5 letters). Prove that $\mathcal{F}_{12} \neq \emptyset$ and $G$ acts transitively on $\mathcal{F}_{12}$. Determine the cardinality of $\mathcal{F}_{12}$, and show that if $K \in \mathcal{F}_{12}$, then $K$ is not a Galois extension of $F$.