DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

September 2003

No aids allowed.
Do all questions.

1. *[20 points]*

a) Define nilpotent for a finite group.

b) Let $D_{12} = \langle r, s \mid r^6 = s^2 = 1,\ rs = sr^{-1} \rangle$ be the dihedral group of order 12. Determine whether D_{12} is a nilpotent group.

2. *[20 points]*

Let R be a commutative ring with identity.

a) Prove that if R is finite, then every nonzero prime ideal of R is a maximal ideal.

b) An ideal I of R is primary if whenever $rs \in I$ and $r \notin I$, then $s^n \in I$ for some positive integer n. Prove that I is primary if and only if every zero divisor in R/I is a nilpotent element.

3. *[20 points]*

Let R be a PID, and let M be a finitely generated R-module of free rank (aka Betti number) 0.

a) State the fundamental theorem giving the invariant factor decomposition of M.

b) Prove that M is a cyclic R-module if and only if M has exactly one invariant factor.

c) Prove that M is a simple (aka irreducible) R-module if and only if M has exactly one invariant factor, and the invariant factor is prime.

4. *[15 points]*
Let F be a field and let $f(x) \in F[x]$. Prove that there exists an extension E of F which contains a root of $f(x)$.

5. *[25 points]*
Let E be a finite Galois extension of a field F, with Galois group $G = \mathrm{Gal}(E/F)$.

a) Let $n = [E : F]$. Let m be a positive integer dividing n, and set $\mathcal{F}_m = \{K \text{ a field} \mid F \subset K \subset E,\ [K : F] = m\}$. Suppose that $\mathcal{F}_m$ is nonempty. Show that the map $G \times \mathcal{F}_m \to \mathcal{F}_m$ given by $(\sigma, K) \mapsto \sigma(K)$, $\sigma \in G$, $K \in \mathcal{F}_m$, defines an action of G on the set $\mathcal{F}_m$. Prove that this action is transitive if and only if any two subgroups of index m in G are conjugate in G.

b) Assume that $G = A_5$ (the alternating group on 5 letters). Prove that $\mathcal{F}_{12} \neq \emptyset$ and G acts transitively on $\mathcal{F}_{12}$. Determine the cardinality of $\mathcal{F}_{12}$, and show that if $K \in \mathcal{F}_{12}$, then K is not a Galois extension of F.