Name: ________________________________

Algebra Qualifying Exam
May, 2004
3 hours
No aids allowed

Answer as many as you can. All questions are of equal value.

1. Let $\sigma \in S_{10}$ be the permutation $\sigma = (123)(456)$. (i) What is the order of $Z_{S_{10}}(\sigma)$, its centralizer in S_{10}?

(ii) Let $H \subset S_{10}$ be the subgroup generated by σ, i.e., $H = \langle \sigma \rangle$. What is the order of $N_{S_{10}}(H)$, the normalizer of H in S_{10}?

2. Suppose p, q, r are distinct primes. If G is a group whose order is $|G| = pqr$, show that G is solvable.

3. What are all the maximal ideals in $\mathbb{R}[x]$?

4. Suppose F is a field. If $f(x) \in F[x]$, describe all the ideals in the quotient ring $F[x]/(f(x))$. Express your answer in terms of the prime factorization $f(x) = p_1(x)^{r_1} \cdot \ldots \cdot p_k(x)^{r_k}$ into positive powers of distinct irreducible factors.

5. Let $G = \mathbb{Z}/3\mathbb{Z}$, and consider the real group ring $A = \mathbb{R}[G]$. Consider the homomorphism $\sigma : G \to M_{2\times 2}(\mathbb{R})$ given by

$$\sigma(n + 3\mathbb{Z}) = \begin{pmatrix} \cos \frac{2\pi n}{3} & \sin \frac{2\pi n}{3} \\ -\sin \frac{2\pi n}{3} & \cos \frac{2\pi n}{3} \end{pmatrix}.$$

(This can be visualized by saying that an element of the cyclic group is represented by a rotation through an appropriate multiple of $120°$). This homomorphism can be extended to an action of A on $V = \mathbb{R}^2$ which makes V into an A-module.

(i) Prove that σ is an irreducible representation of G on $V = \mathbb{R}^2$. This is equivalent to saying that V is a simple A-module.

(ii) Suppose $\sigma_{\mathbb{C}}$ is the representation of G on $V_{\mathbb{C}} = \mathbb{C}^2$ given by the same formula. Show that it is reducible, and find its decomposition into irreducible representations. This is equivalent to decomposing the $\mathbb{C}[G]$-module $V_{\mathbb{C}} = \mathbb{C}^2$ into simple modules.

... continued ...

6. A finite extension K/F is said to be "abelian" if it is a Galois extension and the corresponding Galois group $\mathrm{Gal}(K/F)$ is an abelian group.

Suppose L/F is a finite Galois extension. Prove that there exists a unique maximal abelian extension F^{ab} of F contained in L, i.e., that $F \subset F^{ab} \subset L$ and any abelian extension K of F contained in L must be contained in F^{ab}.

Hint: the Fundamental Theorem of Galois Theory.