Algebra Qualifying Exam
September, 2004
3 hours
No aids allowed

Answer as many as you can. All questions are of equal value.

1. (i) What is the smallest positive integer n so that the symmetric group S_n contains an element of order 18?

(ii) Let $\sigma \in S_{10}$ be the permutation $\sigma = (12)(345)(678)$. What is the order of $Z_{S_{10}}(\sigma)$, its centralizer in S_{10}?

2. (i) Write $(\mathbb{Z}/12\mathbb{Z}) \times (\mathbb{Z}/45\mathbb{Z})$ as a product of cyclic groups, with the order of each factor a multiple of the order of the next, in the usual way.

(ii) Write $(\mathbb{Z}/12\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/45\mathbb{Z})$ as a product of cyclic groups, with the order of each factor a multiple of the order of the next, in the usual way.

3. Suppose p, q, r are distinct primes. If G is a group whose order is $|G| = pqr$, show that G is solvable.

4. Determine all the ideals in the ring $\mathbb{Z}[x]/(2, x^3 + 27)$.

5. (i) Suppose R is a commutative ring with unit. Show that an R-module M is irreducible if and only if it is isomorphic as an R-module to R/I, for some maximal ideal I.

(ii) Let $A = \mathbb{Q}[x]$, regarded as an algebra over $\mathbb{Q}$. Prove or disprove: A is a semisimple algebra over $\mathbb{Q}$.

6. (i) Explain why $f(x) = x^3 + 8x - 6$ is irreducible as an element of $\mathbb{Q}[x]$.

(ii) What is the Galois group of the splitting field of $f(x) = x^3 + 8x - 6$ over $\mathbb{Q}$?