Algebra Qualifying Exam
May 6, 2005
3 hours
No aids allowed

Answer as many as you can. All questions are of equal value.

1. How many Sylow subgroups of each possible order are there in the symmetric group S_4? Are any of them normal?

2. (i) Write $(\mathbb{Z}/12\mathbb{Z}) \times (\mathbb{Z}/30\mathbb{Z})$ as a product of cyclic groups, with the order of each factor a multiple of the order of the next, in the usual way.

(ii) Let $M = \mathrm{Hom}(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/30\mathbb{Z})$, the set of homomorphisms $\phi : \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/30\mathbb{Z}$. The set M is an abelian group under the addition

$$(\phi + \psi)(x) = \phi(x) + \psi(x),$$

for $\phi, \psi \in M$, $x \in \mathbb{Z}/12\mathbb{Z}$, and where the plus sign on the right side refers to addition in $\mathbb{Z}/30\mathbb{Z}$. Write M as a product of cyclic groups, with the order of each factor a multiple of the order of the next, in the usual way.

3. For each of the following, either prove it is true or give a counterexample.

(i) If A and B are real 4×4 matrices with the same characteristic polynomial and the same minimal polynomial, then they have the same Rational Canonical Form.

(ii) If A and B are real 5×5 matrices and, in both cases, the minimal polynomial is $p(t) = t(t-4)^3$, then A and B are similar.

4. (i) Suppose R is a ring with unit. Prove that any quotient of a cyclic (left) R-module is cyclic. *(Recall: a module is "cyclic" if it is generated by a single element.)*

(ii) How many simple modules (up to isomorphism) are there for the group ring $\mathbb{C}[S_4]$, and what are their dimensions?

5. Find the invariant factors and elementary divisors of the transformation T whose matrix relative to some basis is $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

6. (i) Show that $f(x) = x^3 - 5$ is irreducible as an element of $\mathbb{Q}[x]$.

(ii) Show that the Galois group of the splitting field E of $f(x) = x^3 - 5$ over $\mathbb{Q}$ is S_3.

(iii) What is the fixed field in E of the subgroup $A_3 \subset S_3 = \mathrm{Gal}(E/\mathbb{Q})$?