Algebra Qualifying Exam
Thursday, September 8, 2005
1–4 p.m.
No aids allowed
Answer as many as you can. All questions are of equal value.

1. Show that there are no simple groups of order 70.

2. Prove or disprove: every p-group is nilpotent.

3. The dihedral group D_{12} of order 12 acts by rotations and reflections on the regular hexagon. Label the vertices of the hexagon $1, 2, 3, 4, 5, 6$. Let $\mathcal{P}$ be the set of ordered pairs of vertices, so the cardinality of $\mathcal{P}$ is 36. Then D_{12} acts on $\mathcal{P}$ in the obvious way: for $\sigma \in D_{12}$, $1 \leq i, j \leq 6$, $\sigma \cdot (i, j) = (\sigma \cdot i, \sigma \cdot j)$, where $\sigma \cdot i$ is the image of vertex i under the original action of D_{12} on the vertices.

(i) Describe the orbits of D_{12} in $\mathcal{P}$.

(ii) For each of the orbits of D_{12} in $\mathcal{P}$, fix a point in the orbit and find the order of the stabilizer of that point in D_{12}.

4. Suppose that R is a finite commutative ring with unit. Show that every prime ideal is maximal.

5. *For each of the following statements, either prove the statement or provide a counterexample (with an explanation of why it is a counterexample).*

(i) Suppose R is a PID and $I \subset R$ is a proper ideal. Then R/I is a PID.

(ii) Suppose R is a PID and $S \subset R$ is a subring containing the unit element 1. Then S is a PID.

(iii) Every Euclidean domain is a UFD.

6. In $\mathbb{C}[x, y]$, let V be the subspace of all polynomials of degree 2 or less, i.e., the subspace spanned by $1, x, y, x^2, xy, y^2$. Let $T : V \to V$ be defined by $T = \frac{\partial}{\partial x}$. Find the invariant factors and elementary divisors of T.

7. Let $f(x) = x^6 - 1 \in \mathbb{Q}[x]$.

(i) Factor $f(x)$ into a product of irreducible factors over $\mathbb{Q}[x]$.

(ii) What is the splitting field of $f(x)$ over $\mathbb{Q}$ and what is its Galois group?