Name: ______________________________

Algebra Qualifying Exam
September 7, 2006
3 hours
No aids allowed

Answer as many as you can. All questions are of equal value.

1. (i) What is the order of the group of rotations of the cube?

(ii) Prove or disprove: the symmetric group S_5 has exactly one 2-Sylow subgroup.

2. Show that there are no simple groups of order 63.

3. Let $G = S_5$, regarded as the permutations of the set $\{1, 2, 3, 4, 5\}$. Let $H \cong S_4$ be the subgroup of all elements of G which fix the number 5. Then H acts on G by conjugation.

(i) Enumerate the orbits of this action. One possible way to express your answer is to give a list of representatives of the distinct orbits.

(ii) For the above action of $H \cong S_4$ on $G = S_5$, what is the order of the stabilizer of the element $(12)(345) \in S_5$?

4. Suppose R is an integral domain (commutative, with unit). Prove or disprove: for any prime ideal $P \subsetneq R$, the quotient R/P must be an irreducible R-module.

5. *For each of the following statements, either prove the statement or provide a counterexample (with an explanation of why it is a counterexample).*

(i) A nontrivial finite group with no proper subgroups must be a cyclic group whose order is a prime number.

(ii) Every quadratic extension of $\mathbb{Q}$ is Galois.

(iii) If R is a PID, then any subring S of R which contains the identity must also be a PID.

6. In $\mathbb{C}[x, y]$, let V be the subspace of all polynomials of degree 2 or less, i.e., the subspace spanned by $1, x, y, x^2, xy, y^2$. Let $T : V \to V$ be defined by $T = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Find the invariant factors and elementary divisors of T.

7. (i) Let $f(x) = x^3 + x - 2 \in \mathbb{Q}[x]$. What is the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$?

(ii) Let $g(x) = x^3 + 6x^2 + 12x + 5 \in \mathbb{Q}[x]$. What is the Galois group of the splitting field of $g(x)$ over $\mathbb{Q}$?