Prof. Valentin Blomer

University of Toronto
Department of Mathematics

Algebra

Comprehensive Exam, Apr. 10, 2007
Total: 75 points

Name:

1. [6 points] Let R be a ring with unity and M a unitary R-module. Define what it means that
a) M is *free*;
b) M is *projective*.
c) Is there a connection between free and projective modules?

2. [6 points] Let K be a field, R a ring, and n a positive integer.
a) What is an affine algebraic variety $V \subseteq K^n$? What is a radical ideal $\mathfrak{a} \subseteq R$?
b) What is the Zariski topology on the vector space K^n?
c) For an algebraically closed field K, state the bijective correspondence obtained in Hilbert's Nullstellensatz. Explain your notation. You should include a statement about maximal ideals of $K[x_1, \ldots, x_n]$.

3. [6 points] Let G be a finite group, and K an algebraically closed field of characteristic zero.

a) What is the regular representation of G?

b) What is the connection between the regular representation and any irreducible finite-dimensional representation $\rho : G \to \mathrm{Aut}_K(V)$ of G in a K vector space V?

4. [6 points] a) Let $L \supseteq K$ be a field extension. Define what it means that the extension is *normal*; *separable*; *Galois*. What is the Galois group of an algebraic extension L/K?

b) State the main theorem of Galois theory for finite field extensions L/K. You should include a statement about normal subgroups of $\mathrm{Gal}(L/K)$.

5. [12 points] Let $n \geq 2$ be an integer. Show that the finitely generated, unitary $\mathbb{Z}/n\mathbb{Z}$-modules M are *exactly* those of the form $M \cong \bigoplus_{j=1}^{k} \mathbb{Z}/n_j\mathbb{Z}$ for some $k \in \mathbb{N}$ and some integers $n_j \mid n$.

Remarks: Recall that unitary means that $1 \cdot m = m$ for all $m \in M$. Note that $\mathbb{Z}/n\mathbb{Z}$ is not in general a principal ideal domain, so the structure theorem is not applicable here.

6. [12 points] Let G be a group. An abelian group G^* is a *blib group* of G, if there exists a surjectve homomorphism $\phi : G \to G^*$ such that for each homomorphism $\psi : G \to H$ into an abelian group H there exists a homomorphism $\theta : G^* \to H$ with $\psi = \theta\phi$.

a) Show that any two blib groups G_1^*, G_2^* of G are isomorphic.

b) Show that every group G has a blib group. What is a blib group in standard terminology?

7. [15 points] Let R be a Noetherian integral domain (with unity) with the following property: For any two ideals $\mathfrak{a} \subseteq \mathfrak{b} \subseteq R$ there is an ideal $\mathfrak{c} \subseteq R$ such that $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$. Show the following statements (in one or two lines each). For the proof of each part you can assume all previous parts even if you have not completed them.

a) For $0 \neq a \in R$ and two ideals $\mathfrak{b}, \mathfrak{c} \subseteq R$ with $(a)\mathfrak{b} = (a)\mathfrak{c}$ one has $\mathfrak{b} = \mathfrak{c}$.

b) For an ideal $\mathfrak{a} \subseteq R$ and $a \in \mathfrak{a}$, there is an ideal $\mathfrak{d} \subseteq R$ with $(a) = \mathfrak{a}\mathfrak{d}$.

c) For ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \subseteq R$ with $\mathfrak{a} \neq \{0\}$ and $\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ one has $\mathfrak{b} = \mathfrak{c}$.

d) Each prime ideal $\{0\} \neq \mathfrak{p} \subseteq R$ is a maximal ideal.

e) For each ideal $\{0\} \neq \mathfrak{a} \neq R$ there is a prime ideal $\mathfrak{p} \subseteq R$ and an ideal $\mathfrak{a} \subsetneqq \mathfrak{b} \subseteq R$ with $\mathfrak{a} = \mathfrak{p}\mathfrak{b}$.

f) Each ideal $\{0\} \neq \mathfrak{a} \neq R$ is a product of finitely many (not necessarily distinct) prime ideals $\mathfrak{a} = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_r$ for some $r \in \mathbb{N}$. (Use that R is Noetherian.)

g) The factorization in the preceding part is unique up to permutation of the factors.

8. [12 points] Let L/K be a Galois field extension of finite degree $[L : K] = n$. For $a \in L$ let $m_a = \sum_j c_j x^j \in K[x]$ be the minimal polynomial of a over K, and let $\phi_a : L \to L$ be defined by $x \mapsto ax$. The map ϕ_a is a field homomorphism, so in particular a K-vector space homomorphism. Let $N_{L/K}(a) := \det \phi_a \in K$ be the *norm* of a.

a) Assume that $L = K(a)$. Show that

$$N_{L/K}(a) = (-1)^n c_0 = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(a).$$

Conlude that $N_{L/K}(a) = N_{L/K}(\sigma(a))$ for all $\sigma \in \mathrm{Gal}(L/K)$.

b) ["Hilbert 90 Theorem"] Assume that $L = K(a)$ is a Galois extension over K with cyclic Galois group $\mathrm{Gal}(L/K) = \langle \sigma \rangle$, and assume $N_{L/K}(a) = 1$. Show that there exists $b \in L^*$ such that $a = b/\sigma(b)$. *Hint:* Consider the map

$$\Psi := \mathrm{id} + a\sigma + (a\sigma(a))\sigma^2 + \ldots + (a\sigma(a) \cdot \ldots \cdot \sigma^{n-2}(a))\sigma^{n-1} : L \to L$$

Choose any $b := \Psi(c) \neq 0$ (why is this possible?).