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Algebra

Comprehensive Exam, Sept. 5, 2007
Total: 75 points

Name:

Note: No proofs required for questions 1 - 4.

1. [6 points] Let G be a group.
a) State the three Sylow theorems.
b) What is a *composition series* of G? To what extent is it unique?

2. [6 points] Let R be an integral domain.
a) Define *prime element* and *irreducible element*.
b) What is the connection between prime elements and irreducible elements in R?
c) Let $R[x]$ be the polynomial ring over R in one variable. Which of the following statements is correct?

- ☐ R is Euclidean implies $R[x]$ is Euclidean.
- ☐ R is a unique factorization domain implies $R[x]$ is a unique factorization domain.
- ☐ R is a principal ideal domain implies $R[x]$ is a principal ideal domain.

3. [6 points] Let R be a commutative ring with unity.

a) Define the tensor product $M \otimes_R N$ of two R-modules M, N.

b) Let $\mathfrak{p} \subseteq R$ be a prime ideal. What is the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$?

4. [6 points] Let G be a finite group, K an algebraically closed field of characteristic 0, and V a finite-dimensional K-vector space. Let $\rho : G \to \mathrm{Aut}_K(V)$ be a representation.

a) What is the character χ of ρ?

b) Give an explicit expression for the character χ_{reg} of the regular representation.

c) State the orthogonality relations for characters.

5. [12 points] Let G be a group, and H a subgroup of index $n \in \mathbb{N}$. Show that there is a normal subgroup K of G of index at most $n!$ that is contained in H.

6. [10 points] Let R be a unique factorization domain in which every nonzero prime ideal is maximal. Show that every nonzero prime ideal $\mathfrak{p} \subseteq R$ of R is a principal ideal.

7. [14 points] Let R be a principal ideal domain, $\{0\} \neq \mathfrak{a}, \mathfrak{b} \subseteq R$ two nonzero ideals in R. Prove or disprove:

$$\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b}) \cong \mathrm{Hom}_R(R/\mathfrak{b}, R/\mathfrak{a}) \quad \text{(as } R\text{-modules)}.$$

Here $\mathrm{Hom}_R(M, N)$ denotes the set of R-module homomorphisms between the two R-modules M and N. *Suggestion:* One possible approach is to determine explicitly the set $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$.

8. [15 points] Let $f \in \mathbb{Q}[x]$ be a polynomial of degree $n \in \mathbb{N}$. Assume that the Galois group $\mathrm{Gal}(f)$ of f is the symmetric group S_n. Let α be a root of f.

a) Show that f is irreducible. *Suggestion:* Assuming $f = gh$, find an embedding $\mathrm{Gal}(f) \hookrightarrow \mathrm{Gal}(g) \times \mathrm{Gal}(h)$.

b) Let $n \geq 3$. Show that the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.

c) Assume $n \geq 4$, and let $\beta := \alpha^n$. Show $\beta \notin \mathbb{Q}$.