UNIVERSITY OF TORONTO

MAT 1100Y

3 hours Final Exam April 29, 2008

<u>*Instructions*</u>: Answer all questions. Unless noted otherwise, explanation and justification of your answers is expected.

1. (32 marks) a) Show that every group of order 200 has a nontrivial normal subgroup.

b) Make a list (up to isomorphism) of all abelian groups of order 200.

c) Let T be a 4×4 matrix with entries in $\mathbb{C}$ whose minimum polynomial is $(\lambda - 2)(\lambda - 3)$. Make a list of all possibilities for the Jordan normal form of T.

d) Find all the ideals of the ring $\mathbb{Z}[x]/(2, x^3 + 26)$.

2. (12 marks) a) Let T belong to $\mathrm{GL}_k(\mathbb{C})$ such that $T^n = I$ (where I denotes the identity matrix).
Show that T is diagonalizable.

b) Let G be a finite group and let $\rho : G \to GL_k(\mathbb{C})$ be a representation. Show that $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ for all $g \in G$. ($\bar{a}$ denotes the complex conjugate of a.)

3. (16 marks) a) Define (or give a condition equivalent to) Noetherian ring.

b) Show that a Principal Ideal Domain is Noetherian.

c) Show that in a Principal Ideal Domain, every prime ideal is maximal.

d) Let R be an integral domain.
(i) Show that if $x \in R$ is prime, then x is irreducible.
(ii) Give an example to show that x can be irreducible but not prime.

4. (16 marks) a) Define the semidirect product of groups.

b) Find groups H and K such that $A_4 \cong H \rtimes K$. (A_4 denotes the alternating group.)

c) Show that A_4 is solvable.

d) Find elements x and y in A_4 such x and y are conjugate in S_4 but x and y are not conjugate in A_4.

e) Find the character table of A_4.

5. (10 marks) Let $f(x)$ be an irreducible cubic in $\mathbb{Q}[x]$.

a) What are the possibilities for the Galois group of f?

b) Given some particular $f(x)$, describe how you would determine which one is the Galois group of f.

6. (14 marks) Let $F \subset K$ be an extension of fields.

a) Define what it means to say that an element $x \in K$ is algebraic over F.

b) Suppose that $a, b \in K$ are such that $F(a)$ and $F(b)$ are normal separable extensions of F with $[F(a) : F]$ relatively prime to $[F(b) : F]$.
(i) Show that $F(a) \cap F(b) = F$.
(ii) Show that $F(a, b) = F(a + b)$.