ALGEBRA QUALIFYING EXAM
MONDAY, APRIL 27, 2009

Rules:

No aids allowed. Each question costs 10 pts. To get the full mark, answer any 3 questions from the first part, any 4 questions from the second part and any 3 questions from the third part of the test.

1. Groups.

(1) Describe all group homomorphisms from the group A_4 to the multiplicative group of complex numbers. The answer and a short proof are required.
(2) Prove that for an odd prime p the number of Sylow p-subgroups in the group of invertible 2×2-matrices over the finite field with p elements equals $p + 1$. The full proof is required.
(3) Decompose the group of invertible elements in $\mathbb{Z}/120$ into the product of cyclic groups. The answer and a short proof are required.
(4) Formulate a necessary and sufficient condition for an Abelian group to be injective. A precise formulation of the theorem is required.

2. Rings and modules.

(1) Prove that any integral domain with finitely many elements is a field. The full proof is required.
(2) Give the full list of irreducible elements in the ring $\mathbb{Z}[i]$. No proof is required.
(3) Prove that the intersection of all prime ideals in a commutative ring A coincides with the nilradical of A.
(4) Let V be an n-dimensional vector space. Construct a canonical isomorphism between the vector spaces $\Lambda^n(V^*) \otimes \Lambda^k(V)$ and $\Lambda^{n-k}(V^*)$. An explicit construction and a short proof are required.
(5) Formulate the classification theorem for finitely generated modules over a PID in the Invariant Factor Form.

(6) Prove that any square matrix A satisfies the equation $f(A)=0$ where $f(t)$ is the characteristic polynomial of A. The full proof is required.

(7) Find the tensor product of the $\mathbb{C}[t]$-modules $\mathbb{C}[t, t^{-1}]$ and $\mathbb{C}$. Here t acts on the second module by 0. The answer and a short proof are required.

3. Fields and Galois Theory.

(1) Prove that for the three fields $F \subset K \subset L$ we have $[L : F] = [L : K] \cdot [K : F]$. The full proof is required.

(2) Find the splitting field and its degree over $\mathbb{Q}$ for the polynomial $x^4 + x^2 + 1$. The answer and a short proof are required.

(3) Prove that any finite extension of the field $\mathbb{Q}$ is simple. A short proof is required.

(4) Find the number of distinct roots of the polynomial $x^{p^2} + x^p - 2$ in the field $\overline{\mathbb{F}}_p$. The answer and a short proof are required.

(5) Describe the subgroup in $Gal(\mathbb{F}_{p^6}/\mathbb{F}_p)$ corresponding the subfield $\mathbb{F}_p \subset \mathbb{F}_{p^2} \subset \mathbb{F}_{p^6}$ by generators and relations.