DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra exam (3 hours)
September 2009

No aids allowed.
Do as many questions as you can. An expected perfect result is 60 pts or higher.

(1) a) Prove that the quotient of a group G by its center $Z(G)$ is a cyclic group if and only if G is Abelian, i.e. $G = Z(G)$. (5 pts)

b) Classify all groups of order 8. (5 pts)

(2) a) Prove that for a group G its commutant H is the maximal Abelian quotient of G. (5 pts)

b) Consider the symmetric group S_n. Prove that there are exactly 2 different group maps from S_n to the multiplicative group of complex numbers $\mathbb{C}^*$. (5 pts)

(3) Let $\mathbb{F}$ be a finite field with q elements. Consider a finite dimensional vector space $\mathbb{F}^n$ over $\mathbb{F}$. Denote the group of linear automorphisms of $\mathbb{F}^n$ (resp. the group of non-strictly upper triangular automorphisms of $\mathbb{F}^n$) by G (resp. by B).

a) Prove that any element of B is a product of a diagonal element and several elements of the form A_{ij}, $i < j$. Here A_{ij} denotes the elementary matrix with units on the diagonal and the only non-zero non-diagonal entry placed in the box (i, j). (5 pts)

b) Find the number of double cosets of G by B. (10 pts)

(4) Consider the set $\mathcal{N}$ of nilpotent square matrices over $\mathbb{C}$ of the size n. The group $GL(n)$ acts on $\mathcal{N}$ by conjugation. Classify the orbits. (10 pts)

(5) Let A be a commutative ring. Prove that a polynomial $f(x) \in A[x]$ is invertible in $A[x]$ if and only if its constant term is

invertible in A and the rest of the coefficients are nilpotent in A. (15 pts)

(6) Let V be a vector space of dimension n. Consider an invertible linear map $A: V \to V$.

a) Give the definition of the exterior powers $\Lambda^k(V)$. (5 pts)

b) Let $\Lambda^{n-1}(A)$ be the corresponding automorphism of $\Lambda^{n-1}(V)$ (it takes $v_1 \wedge \ldots \wedge v_{n-1}$ to $A(v_1) \wedge \ldots \wedge A(v_{n-1})$). Find the determinant of $\Lambda^{n-1}(A)$ as a function of $det(A)$. (10 pts)

(7) a) Define the n-th cyclotomic polynomial $\Phi_n(x)$ over $\mathbb{Q}$. (5 pts)

b) Find explicitly $\Phi_{15}(x)$. (5 pts)

(8) Determine the splitting field and its degree over $\mathbb{Q}$ for the polynomial $x^6 - 4$. (10 pts)

(9) a) Classify the finite Galois extensions of a finite field $\mathbb{F}_p$. State the description of the Galois groups for the extensions. (5 pts)

b) Let $q = p^n$. Prove that the multiplicative group of the field $\mathbb{F}_q$ is cyclic. (5 pts)

c) Prove that for any m there exists an irreducible polynomial of degree m over $\mathbb{F}_p$. (5 pts)