DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

Thursday, September 8, 2011, 1-4 PM

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

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Good Luck!

Problem 1.

1. Find all groups of order 1001. (Reference material: $1001 = 7 \cdot 11 \cdot 13 = 7 \cdot 143 = 11 \cdot 91 = 13 \cdot 77$.)
2. Determine all the possible orders of elements in the permutation group S_8 and for each possible order write an element with that order. Explain why no other orders are possible.

Problem 2. Let $\mathbb{Z}/n$ denote the cyclic group of order n.

1. Write the product group $(\mathbb{Z}/20) \times (\mathbb{Z}/90)$ as a product of cyclic groups, with the order of each factor dividing the order of the next.
2. Regarding Abelian groups as $\mathbb{Z}$-modules, write the tensor product $(\mathbb{Z}/20) \otimes_{\mathbb{Z}} (\mathbb{Z}/90)$ as a product of cyclic groups, with the order of each factor dividing the order of the next.
3. Let U be the multiplicative group of units in $\mathbb{Z}/60$. Find the order of U and write U as a product of cyclic groups, with the order of each factor dividing the order of the next.

Problem 3. Let R be a ring.

1. Define "R is a PID" and "R is a UFD".
2. Prove that R is a PID if and only if it is a UFD in which $\gcd(a, b) \in \langle a, b\rangle$ for every non-zero $a, b \in R$.

Problem 4. Suppose that F is a field of characteristic zero, $f(x) \in F[x]$ has degree 4, $\alpha \in \overline{F}$ is a root of f, $[F(\alpha) : F] = 4$ and $F(\alpha)$ is not Galois over F.

1. Prove that the Galois group of f over F is S_4, A_4 or the dihedral group D_8 of order 8.
2. Prove that if D_8 is the Galois group of f over F, then $F(\alpha)$ contains a quadratic extension of F.

Problem 5. Let G be a finite group.

1. Let χ be the character of an finite-dimensional complex representation of G. Prove that $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$. (*Hint*: (One possible approach) If χ is the character of the representation ρ, what can you say about the eigenvalues of $\rho(g)$?)
2. Let $g \in G$. Prove that g is conjugate to g^{-1} if and only if $\chi(g) \in \mathbb{R}$ for every irreducible complex-valued character χ of G.

Good Luck!