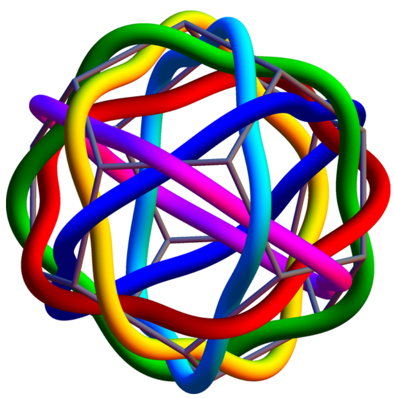
DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

Thursday, September 6, 2012, 1-4 PM

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.



A_5 injects into S_6

Good Luck!

Problem 1. Let K and H be two subgroups of a group G such that $G = KH$ and $K \cap H = \{e\}$.

1. Prove that the multiplication map $m : K \times H \to G$ mapping $(k, h) \mapsto kh$ is a bijection. Is it always a group homomorphism?

2. Prove that the "opposite" multiplication map $\mu : H \times K \to G$ mapping $(h, k) \to hk$ is also a bijection. (Hint: $(ab)^{-1} = b^{-1}a^{-1}$.)

3. Many believe that in a situation as here, G must be a semi-direct product of K and H. Show them wrong by taking $G = S_n$, $H = S_{n-1}$, and an appropriate K, or by any other means.

Problem 2. Let H_1 and H_2 be subgroups of some group G. Prove that the left G-sets G/H_1 and G/H_2 are isomorphic (as left G-sets) iff the subgroups H_1 and H_2 are conjugate.

Problem 3. Let R be a PID and let $a, b \in R$ be such that $\gcd(a, b) = 1$.

1. Prove that there are $s, t \in R$ such that $sa + tb = 1$.

2. Prove that the R-module $R/\langle a \rangle \oplus R/\langle b \rangle$ is isomorphic to the R-module $R/\langle ab \rangle$.

3. Prove that the R-module $R/\langle a \rangle \otimes R/\langle b \rangle$ is isomorphic to the trivial R-module 0.

Problem 4. Let F be a field extension of $\mathbb{Q}$. We will be interested in field maps $\phi : F \to F$ which are not automorphisms.

1. Find an example of such a pair (F, ϕ).

2. Does there exist an example when F is an algebraic extension of $\mathbb{Q}$?

Problem 5. Let k be an algebraically closed field.

1. What does it mean that an affine algebraic variety $X \subset \mathbb{A}^n_k$ is irreducible? Characterize, with proof, irreducible affine algebraic varieties in terms of their ideals.

2. Find the irreducible components of the algebraic variety $X \subset \mathbb{A}^3_k$ defined by the equation $x^2z - y^2z = 0$.

Problem 6. Let k be an algebraically closed field. Consider the representation of S_n on the vector space k^n by permuting the coordinates.

1. Assume that $k = \mathbb{C}$. Give a decomposition of k^n as a direct sum of irreducible subrepresentations. Prove that your summands are irreducible in the case $n = 4$.

2. Assume that the characteristic of k divides n. Prove that k^n contains a subrepresentation which does not have a complementary subrepresentation.

Good Luck!