MAT1100Y Algebra Exam
May, 2006
3 hours
No aids allowed

Explain your answers.
Answer as many as you can. All questions are of equal value.

1. The dihedral group D_{30} is the group of rotations and reflections of the regular 15-gon: $D_{30} = \langle \sigma, \rho | \sigma^2 = 1 = \rho^{15}, \sigma\rho = \rho^{-1}\sigma \rangle$. How many Sylow subgroups of each possible order are there in D_{30}? Are any of them normal?

2. (i) Write $(\mathbb{Z}/20\mathbb{Z}) \times (\mathbb{Z}/90\mathbb{Z})$ as a product of cyclic groups, with the order of each factor a multiple of the order of the next, in the usual way.

(ii) Regarding abelian groups as $\mathbb{Z}$-modules, write $(\mathbb{Z}/20\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/90\mathbb{Z})$ as a product of cyclic groups, with the order of each factor a multiple of the order of the next, in the usual way.

(iii) Consider the abelian group $U = (\mathbb{Z}/60\mathbb{Z})^{\times}$, i.e., the (multiplicative) group of units in $\mathbb{Z}/60\mathbb{Z}$. Find the order of U and write U as a product of cyclic groups, with the order of each factor a multiple of the order of the next, in the usual way.

3. Let $G = \mathbb{Z}/3\mathbb{Z}$, $F = \mathbb{F}_3$, the field of three elements. Show that the group algebra $F[G]$ is not semisimple.

4. Let F be a field, and $A \in M_n(F)$. Use A to make F^n into an $F[x]$-module, by $x \cdot v = Av, \forall v \in F^n$. The matrix A is said to be "semisimple" if F^n is a semisimple $F[x]$-module.

(i) If $F = \mathbb{C}$, show that A is semisimple iff A is diagonalizable.

(ii) Give an example to show that the result of part (i) is not true for all fields.

5. Let $\mathcal{P}_3$ be the real vector space of all polynomials of degree less than or equal to 3. Define a linear operator $T : \mathcal{P}_3 \to \mathcal{P}_3$ by

$$T(f(x)) = (x^2 + x)\frac{d^2 f}{dx^2} - 4x\frac{df}{dx} + 3f(x).$$

Find the invariant factors and elementary divisors of the transformation T.

6. (i) Show that $f(x) = x^3 - 3x - 3$ is irreducible as an element of $\mathbb{Q}[x]$.

(ii) Show that the Galois group of the splitting field E of $f(x) = x^3 - 3x - 3$ over $\mathbb{Q}$ is S_3.

(iii) What is the fixed field in E of the subgroup $A_3 \subset S_3 = \mathrm{Gal}(E/\mathbb{Q})$?