DEPARTMENT OF MATHEMATICS
University of Toronto

Practice exam in Analysis (3 hours)

1. How many roots does the equation $z^8 - 2z^5 + 6z^3 - z + 1 = 0$ have in the region $|z| < 1$?

2. (i) Find one $1-1$ onto conformal map f that sends the open quadrant $\{(x, y) : x > 0 \text{ and } y > 0\}$ onto the open lower half disc $\{(x, y) : x^2 + y^2 < 1 \text{ and } y < 0\}$.

(ii) Find **all** such f.

3. (i) Define almost everywhere convergence and convergence in L_1-norm.

(ii) Show by example that neither form of convergence implies the other.

(iii) Prove that any sequence which is Cauchy in L_1-norm has a subsequence which converges a.e.

4. (i) Define the space $\mathcal{S}$ of Schwartz functions on $\mathbb{R}$.

(ii) State the Fourier inversion theorem.

(iii) Prove that the Fourier transform $f \mapsto \hat{f}$ maps $\mathcal{S}$ onto $\mathcal{S}$.

5. (i) Define the spectrum of a bounded linear operator T on a Hilbert space $\mathcal{H}$.

(ii) What is meant by compactness of such a T?

(iii) If $\mathcal{H}$ has an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and $\{a_n\}$ is a sequence of complex numbers converging to 0 define T by $Te_n = a_n e_n$. Prove directly that T is compact. What is the spectrum of T?

6. (i) What is a tempered distribution on $\mathbb{R}$?

(ii) Define the derivative of a tempered distribution.

(iii) Show that

$$\langle F, f \rangle = \int_{-\infty}^{\infty} \log|x| f(x) dx$$

defines a tempered distribution F and that

$$\langle F', f \rangle = PV \int_{-\infty}^{\infty} \frac{1}{x} f(x) dx \equiv \lim_{\varepsilon \to 0^+} \int_{\{|x| > \varepsilon\}} \frac{1}{x} f(x) dx \ .$$