DEPARTMENT OF MATHEMATICS

University of Toronto

Analysis Exam (3 hours)

January 1995

1. State each of the following carefully and precisely:

(i) The Cauchy integral formula,

(ii) The Riemann mapping theorem,

(iii) The Poisson integral formula,

(iv) Rouché's theorem, and

(v) The Hahn Banach theorem.

2. State the Lebesgue dominated convergence theorem. State a criterion for differentiation under the integral in $\dfrac{d}{dt}\displaystyle\int_X f(x,t)d\mu(x)$.

Use the Lebesgue theorem to prove the criterion.

3. Evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ by computing the Fourier coefficients of the function $f(x) = x$ in $L^2[0,1]$.

4. For $f \in C[0,1]$ define the Lipschitz norm $\|f\| \equiv |f(0)| + \displaystyle\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}$. Let $X = \{f \in C[0,1] : \|f\| < \infty\}$. Prove that $\|\cdot\|$ is a norm on X and that $(X, \|\cdot\|)$ is a Banach space.

5. (i) What is a tempered distribution on $\mathbb{R}$?

(ii) Define the Fourier transform of such a tempered distribution.

(iii) For each of the following tempered distributions F identify $\hat{F}$ as a function.

(a) $\langle F, f \rangle = f(a) \qquad (a \in \mathbb{R})$

(b) $\langle F, f \rangle = \int_a^b f(x)dx$

6. (i) Find one 1-1 onto conformal map f that sends the strip $\{(x, y) : \ x \in \mathbb{R}, \ y \in (0, 1)\}$ onto the open unit disc $\{(x, y) : \ x^2 + y^2 < 1\}$.

(ii) Find **all** such f.