DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

September 1995

1. State each of the following carefully and precisely:

   (a) Liouville's theorem,

   (b) the maximal modulus theorem,

   (c) the Dirichlet problem,

   (d) Fubini's theorem, and

   (e) the Lebesgue dominated convergence theorem.

2. Consider a formal expression $f(z) = \sum_{-\infty}^{\infty} a_n z^n$. What conditions on a_n are **equivalent** to each of the following statements, respectively:

   (a) f is analytic on $\{1 < |z| < 2\}$,

   (b) $f\left(\frac{1}{z}\right)$ has a pole of order 3 at $z = 0$,

   (c) f has an isolated singularity at $z = 0$, and the residue of f at $z = 0$ is 4,

   (d) f is analytic on $\mathbb{C}$ and f sends $\mathbb{C}$ into $\{z : \operatorname{Re} z > 0\}$,

   (e) f restricted to $D = \{z : |z| < 1\}$ is a 1-1 onto analytic function $D \to D$.

3. (a) Suppose u is a finite Borel measure on $\mathbb{R}$ and $f \in C_0(\mathbb{R})$. Evaluate, with proof,

   (i) $\lim\limits_{t\to\infty} \int\limits_{\mathbb{R}} f(tx)d\mu(x)$

   and

   (ii) $\lim\limits_{t\to 0} \int\limits_{\mathbb{R}} f(tx)d\mu(x)$.

   (b) Suppose that f is a bounded, measurable, periodic function on $\mathbb{R}$ of period 1. Evaluate, with proof,

$$\lim_{t\to\infty} \int_0^1 f(tx)dx \ .$$

4. Let X be a complex normed vector space.

   (a) Prove that a linear functional $\varphi\colon X \to \mathbb{C}$ is continuous if and only if its kernel is a closed subspace of X.

   (b) Prove that if Y is a finite-dimensional subspace of X then there is a closed subspace Z of X such that $Y \cap Z = \{0\}$ and $X = Y + Z$.

5. If $f \in L^1(\mathbb{R})$ and $\xi \hat{f}(\xi) \in L^1(\mathbb{R})$ ($\hat{f}$ denotes Fourier transform) prove that there is a $g \in C^1(\mathbb{R})$ such that $g = f$ a.e., and that $g' \in C_0(\mathbb{R})$.

6. Suppose T is a bounded normal operator on a Hilbert space $\mathcal{H}$, $\|T\| \leq 1$ and 1 is not an eigenvalue of T. Prove that

$$\left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i x \right\| \to 0 \quad \forall x \in \mathcal{H} .$$

   (Hint: One way to do this is to use the spectral theorem to represent T as a multiplication operator, thereby converting the summation into a geometric series which can be summed explicitly.)