DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

January 1996

1. State each of the following carefully and precisely:
   (a) The Cauchy Riemann conditions for an analytic function.
   (b) The definition of an essential singularity as an isolated singularity for an analytic function.
   (c) The definition of a Laurent series and the corresponding Laurent's Theorem.
   (d) The little Picard Theorem.
   (e) The spectral theorem for a normal operator.

2. How many zeros does the polynomial

$$f(z) \; = \; z^6 + 5z^5 - 21z + 1$$

have in the annulus $\{1 < |z| < 2\}$? Why?

3. Suppose $h \in C^1[0,1]$ and ν is a finite Borel measure on $[0,1]$. Let $G(x) = \nu[0,x]$. Prove the following integration by parts formula:

$$\int_0^1 h(x)d\nu(x) \; = \; h(1)G(1) - \int_0^1 h'(x)G(x)dx \; .$$

(Hint: Fubini's theorem.)

4. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the usual Legesgue measure and for $f \in L^1(\mathbb{T})$ let $\hat{f}(n)$ denote the nth Fourier coefficient of f. For $f \in L^1(\mathbb{T})$ let

$$|||f||| \; = \; \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$$

and let

$$B \; = \; \{f \in L^1(\mathbb{T}) : \; |||f||| < \infty\} \; .$$

(a) Show that $|||\cdot|||$ is a norm on the complex vector space B, making B into a Banach space.

(b) If $f, g \in L^2(\mathbb{T})$ show that $f * g \in B$.

5. (a) Define the Fourier transform $\widehat{F}$ and the derivative F' of a tempered distribution F. Express $(F')^\wedge$ in terms of $\widehat{F}$.

(b) If δ_a denotes the unit point mass at a ($a \in \mathbb{R}$), find $\hat{\delta}_a$.

(c) If $\Phi(\xi) = \frac{\sin 2\pi\xi}{\xi}$ find the inverse Fourier transform $F = \check{\Phi}$ as a distribution. Observe that F corresponds to a function $\varphi \in L^1$ and explain why $\hat{\varphi} = \Phi$ as functions.
(Hint: Since $\widehat{F} = \Phi$, $(F')^\wedge$ can be expressed in terms of complex exponentials. Then use (b) to find F'.)

(d) Evaluate $\int_{-\infty}^{\infty} \Phi(\xi)^2 d\xi$.

6. Suppose $f_1, f_2, \ldots$ are measurable, positive, finite-valued functions on $[0, 1]$. Show that there exist contants $c_n > 0$ such that $\sum_{n=1}^{\infty} c_n f_n < \infty$ a.e.