DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

September 1996

No aids.
Do all questions.
Questions will be weighted equally.

1. Let X be a Banach Space and let M be a closed subspace of X. (A Banach Space is a complete normed vector space. It may consist only of the zero vector.)
   (a) Show that $\|x+M\| = \inf_{y\in M}\|x+y\|$ defines a norm on X/M, and that X/M is also a Banach Space with this norm.
   (b) Let Y be another Banach Space and let $T : X \to Y$ be a bounded linear operator whose range $Z = TX$ is a closed subspace of Y. Let $M = \{\, x \,|\, Tx = 0 \,\}$ be the kernel of T. Then show that X/M and Z are isomorphic Banach Spaces (that is, they are both Banach Spaces and there is an invertible linear map between them which is continuous in both directions). Also, give an example to show that Z is not always closed.

2. Consider the L_p spaces associated with the unit interval $X = [0,\,1]$, together with its usual Lebesgue measure μ. Let $g \in L_1$, $1 < p < \infty$, and $q = p/(p-1)$. Show that if

$$\left|\int_X fg d\mu\right| \leq \|f\|_p$$

for each step function f, then $g \in L_q$ and $\|g\|_q \leq 1$. (A step function is a (finite) linear combination of the characteristic functions of some *intervals* in X. To show that g is an L_q function is an important part of the problem.)

3. Let $\mathtt{T}$=$\mathtt{R}/\mathtt{Z}$ be the unit circle with Lebesgue measure. Given a sequence $(a_k)_{k\in\mathtt{Z}}$ of complex numbers, let $s_n(x) = \sum_{k=-n}^{n} a_k e^{2\pi i k x}$ and

$$\sigma_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} s_n(x)\,,$$

where $n, N = 0, 1, 2, \ldots$. If $1 < p < \infty$, then show that $\sup_N \|\sigma_N\|_p < \infty$ if and only if $(a_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of a function $f \in L_p$ (that is, $a_k = \hat{f}(k) = \int_{\mathrm{T}} f(t)e^{-2\pi i k t}dt$). What happens if $p = 1$?

4. Let T be the unit circle with Lebesgue measure. Note that the addition in T is the usual addition in R, modulo 1. If f is a complex valued function defined on T, then let $(Sf)(t) = f(t+\alpha)$, where α is a fixed irrational number in T. First verify that S is a normal operator in L_2 (that is, S commutes with its adjoint S^*). Then prove that the cyclic subspace generated by a function $f \in L_2$ is exactly the set of L_2 functions g such that, for each $n \in \mathrm{Z}$, $\hat{g}(n) = 0$ whenever $\hat{f}(n) = 0$. (Here $\hat{g}(n)$ and $\hat{f}(n)$ denote the Fourier coefficients. Recall that the cyclic space generated by f is the L_2-norm closure of the linear span of functions of the form $(S^*)^m S^n f$, where m and n are nonnegative integers.)

5. (a) Suppose that f is an entire function such that $|f(z)| \leq C|z|^{1/2}$ for $|z|$ sufficiently large. (C is a constant.) What can you conclude about the form of f? Give a proof.
   (b) Show that the set of automorphisms of the unit disc consists precisely of the linear fractional transformations of the form

$$e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z}$$

   where $|a| < 1$ and $\theta \in \mathbb{R}$.

6. Evaluate via residues

$$\int_0^\infty \frac{dx}{x^{1/2}(1+x^2)} \ .$$