DEPARTMENT OF MATHEMATICS
University of Toronto

**Analysis Exam (3 hours)**

*January 1997*

No aids.
Do all questions.
Each question is worth 20 marks.

1. A Banach space $X$ is called uniformly convex if $\lim_n \|x_n - y_n\| = 0$ whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in $X$ such that $\|x_n\| = \|y_n\| = 1$ for each $n$ and such that $\lim_n \|x_n + y_n\| = 2$. Let $X$ be a uniformly convex Banach space.

   a) Let $\{x_n\}$ be a sequence in $X$ and $x \in X$ such that $\lim_n \|x_n\| = \|x\|$ and such that $\lim_n f(x_n) = f(x)$ for each $f \in X^*$, the dual space of $X$. Show that $\lim_n \|x - x_n\| = 0$.

   b) Let $K$ be a non-empty closed and convex subset of $X$. ($K$ is convex means that $ax + (1-a)y \in K$ whenever $x \in K$, $y \in K$, and $a$ is a real number, such that $0 \le a \le 1$.) Show that there is a unique vector $x_0 \in K$ such that $\|x_0\| \le \|x\|$ for all $x \in K$.

2. A bi-infinite sequence of vectors $\{x_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is called stationary if $\langle x_n, x_m\rangle = \langle x_{n-m}, x_0\rangle \ \forall\ m, n \in \mathbb{Z}$. Show that $\{x_n\}$ is stationary if and only if there is a unitary operator $U$ on $\mathcal{H}$ such that $x_n = U^n x_0 \ \forall\ n \in \mathbb{Z}$. (Recall that $U$ unitary means $U^*U = UU^* = \mathrm{id}$.) If $\{x_n\}$ is stationary show that there is a Borel measure $\mu$ on the unit circle $\mathbb{T}$ such that

$$\int z^n d\mu \;=\; \langle x_n, x_0\rangle \ .$$

3. Suppose $f \in L^2(\mathbb{T})$ and let $a_n = \hat{f}(n)$, the $n$th Fourier coefficient. Prove that if $f$ is $C^1$ (i.e. continuously differentiable) then $\sum_{n\in\mathbb{Z}} n^2|a_n|^2 < \infty$. Prove that if $\sum_{n\in\mathbb{Z}} |na_n| < \infty$ then $f$ is $C^1$.

4. a) Let $\mu$ be Lebesgue measure on $[0,1]$. Let $\{\varphi_n\} \subset L^1(\mu)$ be an $L^1$-bounded sequence (i.e. $\|\varphi_n\|_1 < C \ \forall n$) and suppose that $\lim\limits_{n\to\infty} \int_{[0,a]} \varphi_n(x)dx = 0$ for each $a \in [0,1]$. Show that $\lim\limits_{n\to\infty} \int f(x)\varphi_n(x)dx = 0$ for each $f \in C[0,1]$.

   b) Give an example of a uniformly bounded sequence $\{\varphi_n\}$ such that

$$\lim_{n\to\infty} \int_{[0,a]} \varphi_n(x)dx = 0 \text{ for each } a \in [0,1],$$

   for which $\lim\limits_{n\to\infty} \int_{[0,1]} |\varphi(x)|dx$ diverges and $\varphi_n(x)$ diverges a.e. with respect to Lebesgue measure.

   c) Show that the conclusion of (a) will not hold if the assumption of $L^1$-boundedness is dropped.

5. Evaluate via residues $\int\limits_{-\infty}^{\infty} \frac{e^{itx}}{x^2+a^2}\, dx$, where $t \in \mathbb{R}$ and $a > 0$.

6. Let $F$ be the family of holomorphic functions on the unit disc $\Delta$ such that $f(0) = 1$ and $\operatorname{Re} f > 0$.

   a) Show that $\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}$ for all $f \in F$ and $z \in \Delta$.

   b) Deduce that $F$ is a normal family.

   c) How large can $|f'(0)|$ be?

Total $= 120$
$120 \times \frac{5}{6} = 100$.