DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

May 1997

No aids.
Do all questions.
Each question is worth 20 marks.

1.

(a) Define what it means for one measure to be absolutely continuous with respect to another.

(b) State the Riesz representation theorem which identifies bounded linear functionals on a Hilbert space.

(c) **Theorem.** *If μ and ν are finite measures on a measurable space $(X, \mathcal{B})$ and $\mu(E) \leq \nu(E) \quad \forall E \in \mathcal{B}$ then there is an $f \in L^1(\nu)$ such that $0 \leq f \leq 1$ $\nu -$ a.e. and $\mu(E) = \int_E f d\nu$ for each $E \in \mathcal{B}$.*

The above theorem is a special case of the Radon-Nikodym theorem. Give a detailed proof the above theorem by defining a linear functional ϕ on $L^2(\nu)$ by $\phi(f) = \int f d\mu$, showing that ϕ is well-defined and bounded and then applying the theorem in (b).

2.

(a) If f is a twice continuously differentiable function on the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ show that $\sum_{n \in \mathbb{Z}} n^4 |\hat{f}(n)|^2 < \infty$, where $\hat{f}(n)$ denotes the n-th Fourier coefficient.

(b) Let f be a twice continuously differentiable function on $\mathbb{T}$ with integral 0 and let α be an irrational number with the property that there is a constant $c > 0$ such that $|\alpha - m/n| > c/n^2$ for all integers m and n. (It is known that such α exist.) Prove that there is a function $g \in L^2(\mathbb{T})$ such that $f(x) = g(x + \alpha) - g(x)$ a.e. (The addition here is of course modulo 1, that is, addition in $\mathbb{T}$.) Hint: write g as a Fourier series with unknown coefficients, solve for the coefficients and show that they are indeed the coefficients of an L^2 function.

3.

(a) State the uniform boundedness principle.

(b) Suppose X is a complex Banach space and $\phi : X \times X \to \mathbb{C}$ is linear and continuous in each argument separately. This means that for each $x \in X$ the functions $\phi_x(y) = \phi(x, y)$ and $\phi^x(y) = \phi(y, x)$ are both linear and continuous on X. Show that ϕ must be jointly continuous, that is, continuous as a function on $X \times X$ Hint: Use the uniform boundedness principle to obtain a constant C such that $|\phi(x, y)| \le C\|x\|\|y\|$, then show that this condition implies continuity.

4. If T is a bounded normal operator on a complex Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ let $Z(x)$ denote the closure of the linear span of $\{T^i x, (T^*)^i x : i = 0, 1, 2, \ldots\}$. Say that the vector x is cyclic for T if $Z(x) = \mathcal{H}$. Also let $W(x)$ denote the linear span (not the closure thereof) of $\{T^i x : i = 0, 1, 2, \ldots\}$.

(a) Let X be a compact subset of the complex plane and μ a finite Borel measure on X. Define T on $L^2(\mu)$ by $(Tf)(z) = zf(z)$. Show that the constant function $1 \in L^2(\mu)$ is cyclic for T .

(b) In the situation of (a) suppose now that $f \in L^2(\mu)$ and $f(z) \neq 0$ for $\mu -$ a.a. z. Show that f is cyclic for T. Hint: in the case where there exist constants c, C such that $0 < c < |f(z)| < C$ you can reduce to the result in (a), and the general case follows easily.

(c) In the situation of (a) show that if T has no eigenvalues then μ is non-atomic, that is, $\mu\{z\} = 0 \quad \forall z \in X$.

(d) Now suppose T is a bounded normal operator on a Hilbert space $\mathcal{H}$ and that T has no eigenvalues. Suppose further that $x, y \in \mathcal{H}$, $y \neq 0$ and that $y \in W(x)$. Use the spectral theorem and the results in parts (b) and (c) to show that $x \in Z(y)$.

5. Suppose that f is holomorphic in the unit disc and $|f(z)| \le \frac{1}{1-|z|^2}$ there. Find the best upper bound you can for $|f^{(k)}(0)|$, $k = 1, 2, \ldots$.

6. Prove Hurwitz' theorem: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of zero-free holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$ which converges uniformly on compact subsets of Ω to a complex-valued function f then either $f \equiv 0$ or f is zero-free.

Total $= 120$
$120 \times \frac{5}{6} = 100$