DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

September 1997

No aids.
Do all questions.
Total $= 120$
$120 \times \frac{5}{6} = 100$

1. *[20 marks]*

How many zeros does $2z^3 - e^{z/2}$ have in the unit disc? Justify your answer.

2. *[20 marks]*

Show that a function which is meromorphic in the extended complex plane must be rational.

3. *[15 marks]*

A measure space $(X, \mathcal{B}, \mu)$ is called **non-atomic** if for all $B \in \mathcal{B}$ such that $\mu(B) > 0$ there exists a $C \in \mathcal{B}$ such that $C \subset B$ and $0 < \mu(C) < \mu(B)$. Prove that if $(X, \mathcal{B}, \mu)$ is non-atomic then the range of μ is $[0, \mu(X)]$.

Suggestion: Say $A, B \in \mathcal{B}$ are equivalent if $\mu(A \Delta B) = 0$ and let $\overline{\mathcal{B}}$ denote the space of equivalence classes. Given $\alpha \in [0, \mu(X)]$ find $B \in \mathcal{B}$ such that $\mu(B) = \alpha$ by applying Zorn's lemma to a suitable subset of $\overline{\mathcal{B}}$. Explain why it is necessary (for this argument) to work in $\overline{\mathcal{B}}$ rather than $\mathcal{B}$.

4. *[15 marks]*

Recall that a Borel measure μ on the Borel σ-algebra $\mathcal{B}$ of a topological space X is **regular** on a set $B \in \mathcal{B}$ if for all $\varepsilon > 0$ there exist a compact set K and an open set U such that $K \subset B \subset U$ and $\mu(U - K) < \varepsilon$. If X is a compact metric space show that any finite Borel measure μ on $(X, \mathcal{B})$ is regular, that is μ is regular on all $B \in \mathcal{B}$.

Suggestion: Start by showing that the class of sets $B \in \mathcal{B}$ on which μ is regular is a σ-algebra.

5. *[20 marks]*

(a) Suppose that X and Y are Banach spaces. For $(x, y) \in X \oplus Y$ (the algebraic direct sum) define $\|(x, y)\|_1 = \|x\| + \|y\|$. Show that $\|\cdot\|_1$ is a norm on $X \oplus Y$ and that $(X \oplus Y, \|\cdot\|_1)$ is a Banach space.

(b) State the open mapping theorem.

(c) Suppose Z is a Banach space and X and Y are closed subspaces such that $X+Y = Z$ and $X \cap Y = \{0\}$. Define the projection $\pi_X \colon Z \to X$ by $\pi_X(z) = x$, where $z = x + y$, $x \in X$, $y \in Y$. Show that π_X is well-defined, linear and continuous.

6. *[30 marks]*

(a) State some form of Fubini's theorem (about integration on product spaces).

(b) Let $\mathbb{T}^2 = [0, 1)^2$ denote the 2-torus, let α be an irrational number in $[0, 1)$ and define a mapping $T : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T(x, y) = (x + \alpha, y + x),$$

the addition being modulo 1. Show that T preserves two-dimensional Lebesgue measure λ on $\mathbb{T}^2$, that is $\lambda\big(T^{-1}(B)\big) = \lambda(B)$ for all Borel subsets B of $\mathbb{T}^2$.

(c) For $f \in L^2(\lambda)$ define $Uf = f \circ T$. Show that U is well-defined on $L^2(\lambda)$ (that is, independent of the choice of representative of f) and that it is a unitary operator from $L^2(\lambda)$ to itself.

(d) Given $m, n \in \mathbb{Z}$ define a function $\psi_{m,n}$ on $\mathbb{T}^2$ by

$$\psi_{m,n}(x, y) = e^{2\pi i(mx+ny)}.$$

Show that the functions $\psi_{m,n}$, $m, n \in \mathbb{Z}$ form an orthonormal basis for $L^2(\lambda)$. Hint: use the Stone-Weierstrass theroem to prove that the $\psi_{m,n}$, $m, n \in \mathbb{Z}$ are complete in $L^2(\lambda)$.

(e) Suppose that $Uf = f$ for some $f \in L^2(\lambda)$. Show that f is a constant function.

Hint: Look at the Fourier series expansion of f, that is, its expansion in terms of the basis in part (d).