DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

May 1998

No aids.
Do all questions.

1. (a) If f is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ when $|z| \leq R/2$.

(b) Suppose f is analytic in a region Ω. Suppose γ is a piecewise smooth closed curve in Ω (not necessarily simple) which does not meet any of the zeros of f. Let p be a nonnegative integer. What is the value of

$$\int_\gamma \frac{f'(z)}{f(z)} z^p dz$$

in terms of the zeros of f?

2. (a) Let f and g be 1-1 analytic mappings from an open connected set $\Omega \subset \mathbb{C}$ onto the unit open disc Δ. Suppose for some point $z_0 \in \Omega$, $f(z_0) = g(z_0) = 0$. What is the relation between f and g?

(b) Let f be a 1-1 analytic map of the unit disc Δ onto the unit square with center 0, satisfying $f(0) = 0$. Show that $f(iz) = if(z)$.

3. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and $h \in L_1(X, \mathcal{F}, \mu)$ be a complex valued function. Show that $\nu(F) = \int_F h d\mu$, $F \in \mathcal{F}$ defines a complex measure on $(X, \mathcal{F})$. Also show that $f \in L_1(X, \mathcal{F}, |\nu|)$, where $|\nu|$ is the total variation of ν, if and only if $fh \in L_1(X, \mathcal{F}, \mu)$.

4. Let c be the set of all convergent real sequences $x = (x_n)$ with the supremum norm $\|x\| = \sup_n |x_n|$. Let c_0 be the subspace of sequences converging to zero.

(a) Show that ℓ_1 is the dual space of c_0.

(b) Find a representation for the dual space of c.

5. Let $\mathbb{T}$ be the unit circle with the Lebesgue measure and let $\mathbb{Z}$ be the set of all integers. Let A be the space of functions f in $L_1(\mathbb{T})$ for which $\|f\|_A = \sum_{k\in\mathbb{Z}} |\hat{f}(k)|$ is finite, where $\hat{f}(k)$ is the k-th Fourier coefficient for f. Show that each function in A is continuous. If f_n is a sequence in A such that f_n converges to a function f uniformly in $\mathbb{T}$ and if $\|f_n\|_A \leq 1$ for all n, then show that $f \in A$ and $\|f\|_A \leq 1$.

6. (a) State the closed graph theorem.

(b) Let Y be a normed subspace of $L^1(\mathbb{R})$ with

$$Y = \left\{ f \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} |xf(x)|dx < \infty \right\}.$$

Let T: $Y \to Y$ be an operator defined as $L(f)(x) = xf(x)$. Show that T is discontinuous but has a closed graph.