DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

September 1998

No aids.
Do all questions.
Questions will be weighted equally.

1. The measure here is the Lebesgue measure on $\mathbb{R}$.

(a) Prove that if $f \in L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R})$, then $f \in L^q(\mathbb{R})$ for all $p_1 \leq q \leq p_2$.

(b) Produce a function f such that $f \in L^p(\mathbb{R})$ **only** when $p = 2$.

2. (a) State the definition of a bounded (continuous) linear operator between two Banach spaces.

(b) Prove that the kernel of a bounded operator is closed.

(c) Prove that, if the kernel is closed, a linear functional is bounded.

3. (a) State Parseval's formula for Fourier Series.

(b) Prove that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. Hint. Use the Fourier series expansion for $f(x) = x$.

(c) Let f be such that

$$|\hat{f}(n)| \leq C|n|^{-k}$$

for all $k > 0$. Prove that $f \in C^\infty$.

4. (a) State the uniform boundedness theorem (Banach Steinhaus) on Banach Spaces.

(b) Let f be measurable such that $f \cdot g \in L^1$ for all $g \in L^q$. Show that $f \in L^p$ $(1/p + 1/q = 1)$.

5. (a) Give a complex-analytic proof of the fundamental theorem of algebra: every nonconstant holomorphic polynomial with complex coefficients has a root.

(b) Prove the Casorati-Weierstrass theorem: Suppose f is analytic in the set $0 < |z-a| < R$ and has an essential singularity at a. Then the range of f is dense in $\mathbb{C}$.

6. Verify that

$$w = \frac{(1+z^n)^2 - i(1-z^n)^2}{(1+z^n)^2 + i(1-z^n)^2}$$

transforms the circular sector $0 < |z| < 1$, $0 < \operatorname{Arg} z < \frac{\pi}{n}$ onto the unit disc.